PHYSICS 210

OVERVIEW OF FINITE DIFFERENCE APPROXIMATION

Discretization

- In numerical analysis one can often approximately solve continuum systems—typically differential equations—through a process known as **discretization**

- In the continuum case, the unknown function(s), for example, will typically be defined on some interval $0 \le t \le t_{\max}$ of the real number line and will thus constitute an infinite number of values (the same infinity as that associated with the entire real number line, or any interval thereof)

- In the discrete case, the unknown function will typically be defined only at a finite (or at most countable, i.e. having the infinity of the integers) number of values $t_n,\ n = 1, 2, \ldots, n_t$

Discretization (continued)

- **1st FUNDAMENTAL PURPOSE OF DISCRETIZATION**
  - Reduce infinite number of “degrees of freedom” to finite number
- **WHY?**
  - Computational resources are finite
- **2nd FUNDAMENTAL PURPOSE OF DISCRETIZATION**
  - Replace differential equations with algebraic equations
- **WHY?**
  - Can solve algebraic equations (linear or non-linear) computationally

Finite Difference Approximation

- Finite difference approximation (**FDA**) is one specific approach to the discretization of continuum systems such as differential equations

- We choose to focus on it here for several reasons
  - Accessibility (requires a minimum of mathematical background)
  - Generality (can be applied to virtually any system of differential equations)
  - Simplicity (relatively easy to apply in many cases)
  - Sufficiency (for many problems, produces results of acceptable accuracy with reasonable computational cost)

- Other important approaches that we will not discuss
  - **Finite element approximation**
  - **Spectral approximation**

Finite Difference Approximation (continued)

- **BASIC IDEA**
  - Derivatives are replaced with algebraic "difference quotients", very similar in spirit to algebraic expressions that are encountered in the standard definition of a derivative in calculus

$$\frac{df(x)}{dx} \equiv f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - In the above

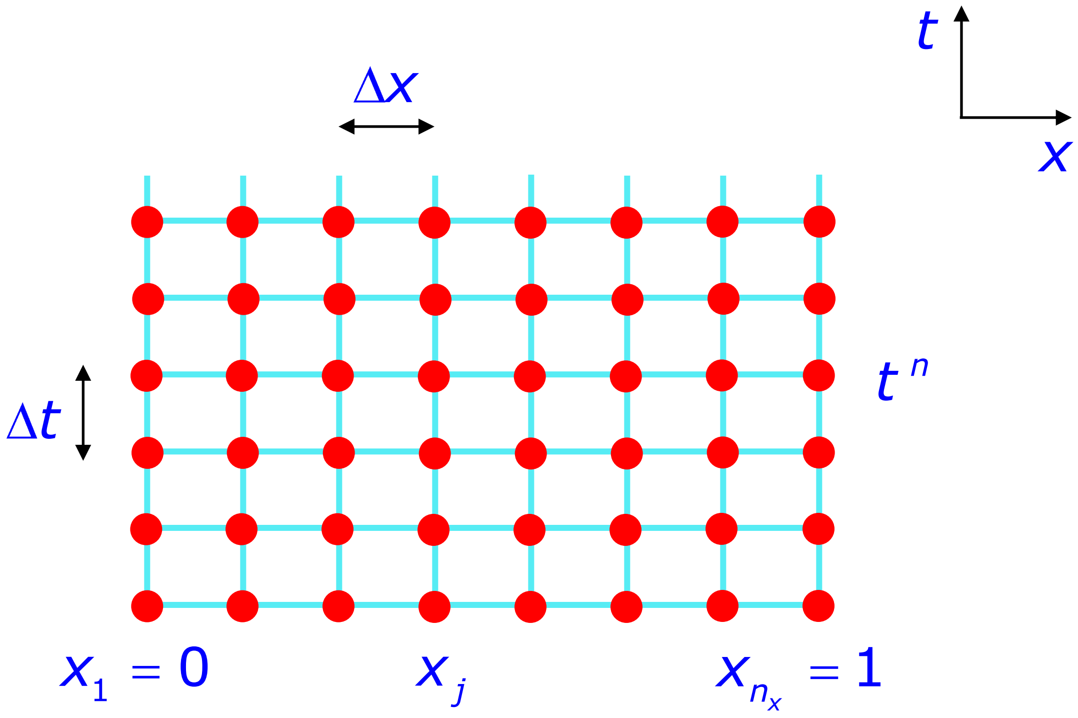
$$\frac{f(x+h) - f(x)}{h}$$

is a finite difference approximation of $f'(x)$

Key Steps in Solution of Differential System Using FDA

1. Formulate **precise** and **complete** mathematical description of the problem to solve, including
   - Specification of independent variables (coordinates)
     - $t,\ x,\ (t,x),\ (t,x,y),\ \ldots$
   - Specification of solution domain in terms of these independent variables
     - $0 \le t \le t_{\max},\ [0 \le x \le 1,\ 0 \le t \le t_{\max}],\ \ldots$
   - Specification of dependent variables and their type (e.g. scalar or vector, real or complex …)
     - $u(t),\ f(x),\ \psi(t,x),\ u(x,y),\ \vec{r}_i(t),\ \ldots$
   - Specification of differential equations governing dependent variables (for time dependent problems, will often call these the equations of motion)
   - Specification of sufficient initial and/or boundary conditions to ensure that the problem has a unique solution.

Schematic of Typical Uniform Finite Difference Mesh



Key Steps in Solution of Differential System Using FDA

2. Discretization: Step 1

   – Define finite difference **grid** (**mesh**, **lattice**) that replaces continuum solution domain with **finite** set of grid points at which discrete solution is to be computed

   – Mesh will be characterized be a set of spacings between adjacent points in each of the coordinate directions; in this course will typically assume that these are constants (so meshes will be called **uniform**)

   – Mesh spacings constitute fundamental parameters that control accuracy of particular FDA

   – Working assumption is that in the limit that the spacings tend to 0, the finite difference solution will **converge** to the continuum solution

Key Steps in Solution of Differential System Using FDA

3. Discretization: Step 2

   – Replace all derivatives—including any involved in the initial or boundary conditions—with finite difference approximations
   – This process yields a set of algebraic equations (linear or non-linear) for the discrete unknowns

4. Solution of algebraic equations

   – The solution of the algebraic equations is then accomplished computationally
   – Depending on the nature of the differential equations as well as the FDA used the sophistication/complexity of the algorithms required to do this efficiently can vary widely

Key Steps in Solution of Differential System Using FDA

5. Convergence testing / error analysis

   - Extremely important part of solution process (difficult to overemphasize importance)

   - Basic idea is to repeat calculations using same basic problem parameters, initial data, boundary conditions etc., but with varying mesh sizes (grid spacings)

   - Investigation of behaviour of finite difference solution as a function of mesh size allows us to estimate (and ultimately control) the accuracy of the solution, and to establish that the solution **is** converging to the desired continuum limit

Important!

- For generality, our discussion so far has largely been in terms of partial differential equations (PDEs), but most of our discussion (and your term projects) will involve the approximate solution of ordinary differential equations (ODEs)

- For example, in the *N*-body problem, or other scenarios dealing with particle dynamics, Newton's laws will result in second-order ODEs in time

- An appropriate FD mesh will then be as illustrated here