

PHY329 WAVE EQUATION (note correction) OCT 23, 1996

GENERATING INITIAL DATA: \hat{u}_j^0 and \hat{u}_j^1

First note a slight change in notation from the original lecture notes. Here, for ease of exposition, \hat{u}_j^n is the quantity which satisfies the difference equation (22.4), while u_j^n satisfies the differential equation (22.2).

We begin by assuming that our difference solution will admit a Richardson expansion

$$\hat{u}(x, t) = u(x, t) + h^2 e_2(x, t) + h^4 e_4(x, t) + \dots \quad (22.7)$$

and ask what this implies about how accurately we must specify \hat{u}_j^0 and \hat{u}_j^1 .

First recall that the problem prescription includes the initial conditions:

$$u(x, 0) = f(x) \quad u_t(x, 0) = g(x)$$

Clearly then, we can (and might as well) specify \hat{u}_j^0 exactly

$$\hat{u}_j^0 = f_j$$

which implies that $e_2(x, 0) = 0$. Then in order for (22.7) to hold, we must have

$$\begin{aligned} \hat{u}_j^1 &\equiv \hat{u}(x_j, \Delta t) \\ &= u(x_j, \Delta t) + h^2 e_2(x_j, \Delta t) + O(h^4) \end{aligned}$$

which implies that

$$\hat{u}(x_j, \Delta t) - u(x_j, \Delta t) = h^2 e_2(x_j, \Delta t) + O(h^4)$$

But

$$\begin{aligned} e_2(x_j, \Delta t) &= e_2(x_j, 0) + \Delta t \frac{\partial e_2}{\partial t}(x_j, 0) + O(\Delta t^2) \\ &= \Delta t \frac{\partial e_2}{\partial t}(x_j, 0) + O(\Delta t^2) = \lambda h \frac{\partial e_2}{\partial t}(x_j, 0) + O(\Delta t^2) \end{aligned}$$

Thus,

$$\hat{u}(x_j, \Delta t) - u(x_j, \Delta t) = h^2 \left(\lambda h \frac{\partial e_2}{\partial t}(x_j, 0) \right) + \dots = O(h^3)$$

So we see that we must specify \hat{u}_j^1 to $O(h^3)$ accuracy, *not* $O(h^2)$ as was erroneously claimed in the lecture. However, this is readily done by simply Taylor-expanding to higher order:

$$u(x_j, \Delta t) = u(x_j, 0) + \Delta t u_t(x_j, 0) + \frac{1}{2} \Delta t^2 u_{tt}(x_j, 0) + O(\Delta t^3)$$

Recalling (22.6) where we specified the initial conditions in terms of initially left-moving and right-moving profiles, $l(x)$ and $r(x)$, and their derivatives, $l'(x)$ and $r'(x)$:

$$\begin{aligned} u(x, 0) &= l(x) + r(x) \\ u_t(x, 0) &= l'(x) - r'(x) \end{aligned}$$

and noting that

$$u_{tt}(x, 0) = l''(x) + r''(x)$$

we have for our initialization:

$$\hat{u}_j^0 = l_j + r_j \tag{22.7}$$

$$\hat{u}_j^1 = l_j + r_j + \Delta t \left(l'_j - r'_j \right) + \frac{1}{2} \Delta t^2 \left(l''_j + r''_j \right) \tag{22.8}$$