THE 3+1 (ADM / HAMILTONIAN) FORM of GR

REFERENCES

(1) MTW CHAPTER 21

(2) YORK, "KINEMATICS & DYNAMICS of GR" IN "SOURCES of GRAV. RADIATION" (L. SMARR ed)

(3) WALD, APP. E.2 ; CHAPTER 10

(4) ARNOWITT, DESER & MISNER (1962) "THE DYNAMICS of GR" IN "GRAVITATION: AN INTRODUCTION TO CURRENT RESEARCH" (L. WITTEN ed)

- ADM MOTIVATION WAS, AGAIN, PREPARATION FOR QUANTIZATION of GR; FORMALISM TURNED OUT TO BE EXCELLENT BASIS FOR COMPUTATIONAL (NUMERICAL) ASSAULT ON EINSTEIN EQUATIONS

APPROACHES

(1) "COORDINATE-FUL" (MTW): INTUITIVE, CONNECTS MORE DIRECTLY TO FORM of E.O.M USED IN PRACTICE

(2) "COORDINATE-FREE" (YORK / WALD): PREFERABLE FOR DERIVATION of E.O.M.

KEY POINT: MUST INTRODUCE A COORDINATE SYSTEM TO NUMERICALLY GENERATE A SPACE-TIME; I.E. CAN'T STAY COORDINATE FREE FOREVER

- WILL ~~START~~ WITH COORDINATE BASED APPROACH TO INTRODUCE CONCEPTS, THEN WILL GO OVER TO COORDINATE-FREE APPROACH TO DERIVE 3+1 EQN'S

ULTIMATE GOAL: REFORMULATE

$$G_{ab} = 8\pi T_{ab}$$

AS SYSTEM of FIRST-ORDER (IN TIME) PDE'S FOR THE GRAV. FIELD VBL^S, WHICH CAN THEN BE SOLVED AS AN "INITIAL-VALUE" or "CAUCHY" PROBLEM

SHIFT IN PERSPECTIVE: UP TO NOW $G_{ab} = 8\pi T_{ab}$ DESCRIBED THE LINKAGE OF THE GEOMETRY of SPACETIME (4-D) TO THE DISTRIBUTION of MATTER-STRESS-ENERGY IN S.T.
NEW VIEW: GEOMETRY of S.T IS "TIME-HISTORY" (EVOLUTION) of GEOMETRY of A SPACELIKE HYPERSURFACE (3-D) => GEOMETRODYNAMICS; "VIEW" NOT UNIQUE SINCE THERE ARE ∞'LY MANY WAYS of "SLICING UP" GIVEN S.T. INTO A FAMILY OF S.L. HYPERSURFACES; NONE PREFERRED (I.E. PHYSICALLY MORE RELEVANT) IN GENERAL

SPLITTING SPACETIME INTO SPACE-PLUS-TIME (THE 3+1 SPLIT)
- SPACETIME IS 4-DIMENSIONAL MANIFOLD M, WITH LORENTZIAN SIGNATURE METRIC g_{ab} $(-+++)$
- INTRODUCE COORDINATES $\{x^\mu\} = \{t, x^i\}$ (MAY NOT COVER ENTIRE S.T., BUT WILL GENERALLY PRETEND THEY DO)

NOTATION/CONVENTIONS:

GREEK INDICES (μ, ν etc.) 0, 1, 2, 3
"INTEGER" LATIN " (i, j, k, l, m, n) 1, 2, 3 (SPATIAL)
WILL ADOPT USUAL EINSTEIN SUMMATION CONVENTION FOR BOTH TYPES!

- DEMAND THAT t = CONST SURFACES (HYPERSURFACES), $\Sigma(t)$, ARE SPACELIKE; I.E. IF DISTINCT EVENTS P, P'

HAVE COORDS (t, x^i), $(t, x^{i'})$ THEN $ds^2(P, P') > 0$

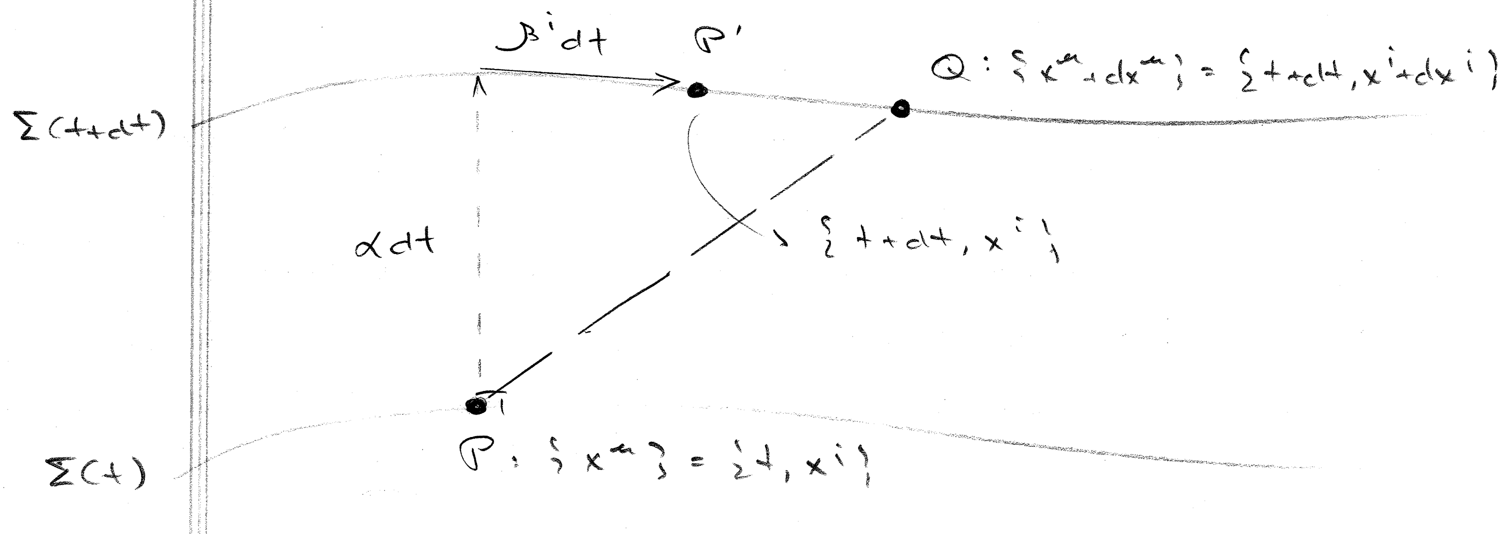
- SAFEST TO VIEW t AS TIME PARAMETER; MAY NOT (IN FACT, IN GENERAL WILL NOT) HAVE ANY PARTIC. SIGNIFICANCE AS A PHYSICAL TIME - SUCH A NOTION IS LARGELY MEANINGLESS IN GR

- EACH $\Sigma(t)$ IS A DIFFERENTIABLE MANIFOLD IN ITS OWN RIGHT, WITH A 3-METRIC ${}^{(3)}g_{ij} = {}^{(3)}g_{(ij)}$ WHICH IS INDUCED ON $\Sigma(t)$ BY THE 4-METRIC ${}^{(4)}g_{\mu\nu}$ OF THE ENVELOPING SPACETIME

- ANY GIVEN FOLIATION ($\equiv$ CHOICE of TIME COORD $\equiv$ CHOICE of SLICING) ALSO DEFINES A NATURAL, UNIT-NORM VECTOR FIELD, n^μ, WHICH IS NORMAL (ORTHOGONAL) TO THE SLICES, AND POINTS "TO THE FUTURE"

$$n^\mu n_\mu = {}^{(4)}g_{\mu\nu} n^\mu n^\nu = -1 \qquad \hat{n}\S\S$$

SPACETIME DISPLACEMENT IN THE 3+1 SPLIT



"SPACETIME PYTHAGOREAN THM" ($\lim dt \to 0$) $\Rightarrow$

$$\text{distance}(P,Q)^2 = {}^{(4)}ds^2$$

$$= -\alpha^2 dt^2 + {}^{(3)}g_{ij}\,(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$= (-\alpha^2 + {}^{(3)}g_{ij}\beta^i\beta^j)\,dt^2 + 2\,{}^{(3)}g_{ij}\beta^i dx^j dt$$

$$+ {}^{(3)}g_{ij}\,dx^i dx^j \qquad (1)$$

NOMENCLATURE:

$\alpha \equiv \alpha(t, x^i)$: LAPSE FUNCTION ("LAPSE") : GIVES LAPSE OF PROPER TIME PER UNIT COORDINATE TIME FOR AN OBSERVER MOVING NORMAL TO THE SLICES

$\beta^j = \beta^j(t, x^i)$: SHIFT VECTOR ("SHIFT") : 3-VECTOR DESCRIBING SHIFTING OF SPATIAL COORDINATES RELATIVE TO "NORMAL PROPAGATION"

- TOGETHER $\{\alpha, \beta^i\}$ CONSTITUTE 4-FOLD COORD. FREEDOM of GR

DUAL VIEWS

(1) α, β^i ARE ESSENTIALLY FREELY SPECIFIABLE FCNS (COORDINATE FREEDOM)

(2) SOME PRESCRIPTION FOR α, β^i MUST BE GIVEN "FROM OUTSIDE" - I.E. E.O.M. (EINSTEIN EQ^N) ALONE WILL NOT, IN GENERAL, DETERMINE THEM ("GAUGE FIXING")

NOTE: TENSORS SUCH AS β^i ARE DEFINED ON $\Sigma(t)$, AND ARE CALLED SPATIAL TENSORS

CLEARLY, ${}^{(3)}g_{ij}$ IS A SPATIAL TENSOR; IT HAS AN ASSOCIATED INVERSE ${}^{(3)}g^{ij}$ SATISFYING

$$ {}^{(3)}g^{ij}\,{}^{(3)}g_{jk} = \delta^i{}_k $$

INDICES ON 3-TENSORS ARE RAISED / LOWERED WITH ${}^{(3)}g^{ij}$, ${}^{(3)}g_{ij}$; THUS, WE CAN REWRITE (1) AS

$$ {}^{(4)}ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_j\,dx^j\,dt + {}^{(3)}g_{ij}\,dx^i dx^j $$

SO, WE HAVE

$$ {}^{(4)}g_{\mu\nu} = \begin{bmatrix} {}^{(4)}g_{00} & {}^{(4)}g_{0j} \\ {}^{(4)}g_{i0} & {}^{(4)}g_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha^2 + \beta^k\beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad (2) $$

IN PARTICULAR, NOTE THAT

$$ {}^{(4)}g_{ij} = {}^{(3)}g_{ij} $$

I.E. SPATIAL COVARIANT COMPONENTS OF 4- AND 3-METRICS ARE IDENTICAL

THIS IS A GENERAL RESULT; GIVEN ANY 4-TENSOR OF TYPE (0, k) (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE COMPONENTS OF A TYPE (0, k) 3-TENSOR

WHY? RECALL THAT COVARIANT TENSOR COMPONENTS CAN BE DEFINED IN TERMS OF THE ACTION OF THE TENSOR ON THE COORDINATE BASIS VECTORS $^{(4)}\vec{e}_\mu$, $\mu = 0, 1, 2, 3$

$$\text{E.G.} \quad {}^{(4)}t_{\mu\nu} = {}^{(4)}t\left({}^{(4)}\vec{e}_\mu, {}^{(4)}\vec{e}_\nu\right)$$

$$\text{AND} \quad {}^{(4)}t_{ij} = {}^{(4)}t\left({}^{(4)}\vec{e}_i, {}^{(4)}\vec{e}_j\right)$$

BUT CLEARLY THE $\{{}^{(4)}\vec{e}_i\}$ ARE PRECISELY THE COORDINATE BASIS VECTORS $\{{}^{(3)}\vec{e}_i\}$ FOR $\Sigma(t)$ WITH COORDINATES $\{x^i\}$; THUS INTERPRETING $^{(4)}t(\ldots)$ AS A 3-TENSOR $^{(3)}t(\ldots)$, WE NECESSARILY HAVE

$$^{(3)}t_{ij} = {}^{(4)}t_{ij}$$

ON THE OTHER HAND, THE SPATIAL MEMBERS $\{{}^{(4)}\underset{\sim}{\omega}^i\}$ OF THE DUAL BASIS $\{{}^{(4)}\underset{\sim}{\omega}^\mu\}$, WILL NOT, IN GENERAL COINCIDE WITH $\{{}^{(3)}\underset{\sim}{\omega}^i\}$ — THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

- THUS, IN GENERAL

$$^{(3)}t^{ij} \neq {}^{(4)}t^{ij}$$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \quad (3)$$

(EXERCISE: VERIFY)

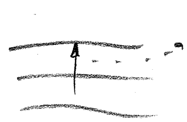
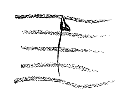
• FROM (2) WE CAN ALSO COMPUTE THE USEFUL RESULT

$$\sqrt{-{}^{(4)}g} = \alpha\sqrt{{}^{(3)}g} \qquad (4)$$

THE NORMAL VECTOR FIELD n^μ

• EASIEST TO START WITH ASSOC. DUAL-VECTOR (ONE-FORM) FIELD, n_μ

GEOM. INTERP of DUAL-VECTOR FIELD: LEVEL SURFACES of SCALAR FUNCTION → DUAL NOTION TO "INFINITESIMAL DISPLACEMENT" (VECTOR)

$\underline{d}f$  $\vec{v}$  $2\,\underline{d}f$

$$\langle \vec{v}, \underline{d}f \rangle = v^\mu (df)_\mu = \#\text{ of level surfaces "pierced" by } \vec{v}$$

• HERE, OUR SCALAR FUNCTION IS THE TIME COORDINATE t, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{d}t$, THEN

$$\underline{n} \propto \underline{d}t$$

OR IN COMPONENT FORM

$$n_\mu = (n_0, 0, 0, 0)$$

THEN, FROM

$${}^{(4)}g^{\mu\nu} n_\mu n_\nu = -1$$

WE HAVE

SIGN CHOSEN SO THAT n^μ IS "FUTURE-DIRECTED"

$$n_\mu = (-\alpha, 0, 0, 0) \qquad (5)$$

AND THEN

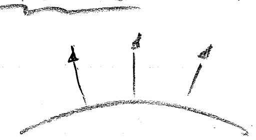
$$n^\mu = {}^{(4)}g^{\mu\nu} n_\nu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right) \qquad (6)$$

EXTRINSIC CURVATURE

- THE INTRINSIC GEOMETRY OF $\Sigma(t)$ IS DESCRIBED BY ${}^{(3)}g_{ij}$ WHICH ENCODES ALL GEOM. INFO. WHICH MAY BE DEDUCED BY MAKING MEASUREMENTS ON $\Sigma(t)$ ALONE

- HOWEVER, A GIVEN 3-GEOMETRY (SLICE, HYPERSURFACE) MAY BE EMBEDDED IN SPACETIME IN INFINITELY MANY DISTINGUISHABLE (BY 4-D MEASUREMENTS) WAYS

EXAMPLE: 2D EMBEDDED IN 3D; FLAT SURFACE EMBEDDED WITH / WITHOUT EXTRINSIC CURVATURE



- THE MANNER IN WHICH $\Sigma(t)$ IS EMBEDDED CAN BE CHARACTERIZED BY INVESTIGATING THE CHANGE IN THE DIRECTION OF THE NORMAL FIELD AS A FCN OF POS ON $\Sigma(t)$ — THIS DEFINES THE EXTRINSIC CURVATURE TENSOR (AKA THE SECOND FUNDAMENTAL FORM)

$$K_{ij} = -\nabla_i n_j = -\nabla_{(i} n_{j)} \qquad (7)$$

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_{\mu}$$

BUT $n_{\mu} = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha\, {}^{(4)}\Gamma^{0}{}_{ij}$$

$$= \alpha \left({}^{(4)}g^{00}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0\ell}\, {}^{(4)}\Gamma_{\ell ij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2} \left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - \underline{{}^{(4)}g_{ij,0}} \right)$$

THUS

$$K_{ij} = -\frac{1}{2\alpha} \frac{\partial\, {}^{(3)}g_{ij}}{\partial t} + \ldots$$

SO WE CAN VIEW THE EXTRINSIC CURVATURE AS THE "VELOCITY" OF THE 3-METRIC ${}^{(3)}g_{ij}$ ("CONJUGATE MOMENTA")

- INTRODUCE COORDS.

$$\{x^\mu\} = \{t, x^i\}$$

MAY NOT COVER ENTIRE S.T.
BUT WILL GENERALLY PRETEND
they do

- NOTATION/CONVENTIONS

GREEK INDICES: (μ, ν, etc.): 0, 1, 2, 3
LATIN INDICES: ($i, j, k, l, m, n \ldots$): 1, 2, 3 (SPATIAL)
(FORTRAN "INTEGER")

WILL ADOPT USUAL EINSTEIN SUMMATION CONVENTION
FOR BOTH TYPES

- REQUIRE t = const surfaces (hypersurfaces)
$\Sigma(t)$ TO BE spacelike; I.E. IF DISTINCT EVENTS
P, P' HAVE coords (t, x^i), $(t, x^{i\prime})$ THEN
$ds^2(P, P') > 0$

- SAFEST TO VIEW t AS A TIME "$\underbrace{\text{PARAMETER}}$";
MAY NOT (IN FACT, IN GENERAL <u>WILL NOT</u>) HAVE
ANY PART. SIGNIFICANCE AS A "PHYSICAL" TIME —
SUCH NOTION LARGELY MEANINGLESS IN GR

• EACH $\Sigma(t)$ IS A DIFF MANIFOLD IN ITS OWN RIGHT
WITH A 3-METRIC ${}^{(3)}g_{ij} = {}^{(3)}g_{(ij)}$ <u>INDUCED</u> ON $\Sigma(t)$
BY THE 4-METRIC ${}^{(4)}g_{\mu\nu}$ OF THE ENVELOPING SPACETIME

THE 3+1 FORMULATION of GR

REFERENCES

1) MTW, CHAPTER 21
2) YORK, IN SMARR VOL (SEE NOTES)
3) WALD, APP E.2 & CHAPTER 10
4) ARNOWITT, DESER & MISNER (1962) "THE DYNAMICS of GR" IN ...

MOTIVATION

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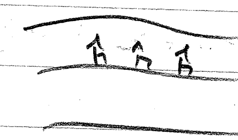

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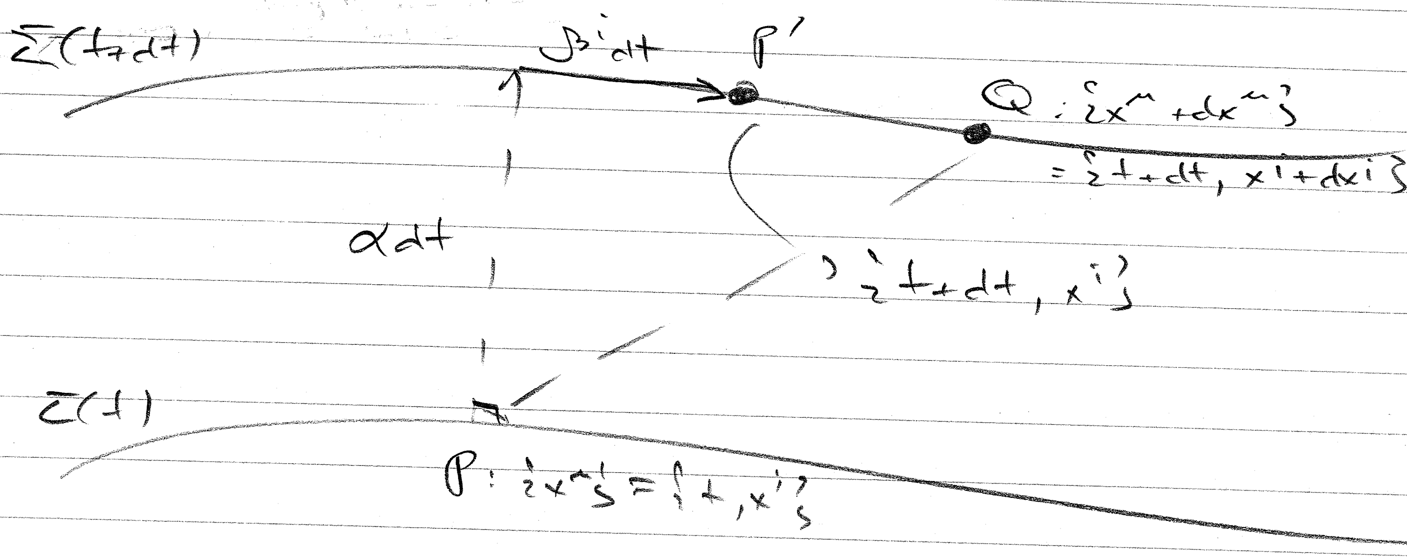

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SO WE HAVE

$$ {}^{(4)}g_{\mu\nu} = \begin{array}{cc} \begin{array}{cc} 1 & 3 \end{array} & \\ \left[\begin{array}{c|c} {}^{(4)}g_{00} & {}^{(4)}g_{0j} \\ \hline {}^{(4)}g_{i0} & {}^{(4)}g_{ij} \end{array}\right] & \begin{array}{c} 1 \\ 3 \end{array} \end{array} = \begin{bmatrix} -\alpha^2 + \beta^k\beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} $$

IN PARTICULAR, NOTE THAT

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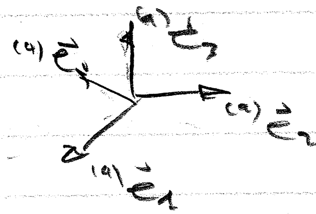
$$^{(4)}\vec{e}_\mu \,, \quad \mu = 0, 1, 2, 3$$

E.G. $\quad {}^{(4)}t_{\mu\nu} = {}^{(4)}t\left({}^{(4)}\vec{e}_\mu, {}^{(4)}\vec{e}_\nu\right)$

AND $\quad {}^{(4)}t_{ij} = {}^{(4)}t\left({}^{(4)}\vec{e}_i, {}^{(4)}\vec{e}_j\right)$



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EXERCISE: VERIFY

FROM (2) CAN ALSO COMPUTE THE USEFUL RESULT.

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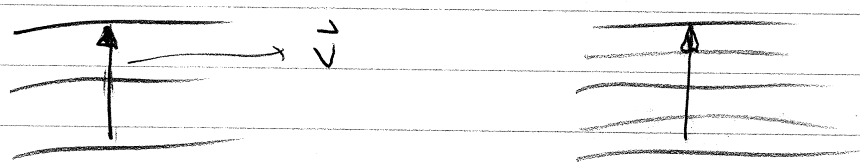
VERIFY

THE NORMAL VECTOR FIELD n^{μ}

EASIEST TO START W. ASSOC. DUAL-VECTOR (ONE-FORM) FIELD, n_{μ}

GEOM. INTERP of DUAL VECTOR FIELD: LEVEL SURFACES of SCALAR FUNCTION → DUAL NOTION to 'INFINITESIMAL DISPLACEMENT' (VECTOR)

$\underline{d}f$



$2\underline{d}f$

$$ \langle \vec{v}, \underline{d}f \rangle = v^{\mu}(\underline{d}f)_{\mu} = \# \text{ of level surfaces "pierced" by } \vec{v} $$

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$$\underline{n} \propto \underline{d}t$$

OR IN COMPONENT FORM

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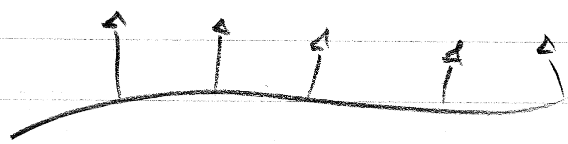

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$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij}\, n_\mu$$

But $n_\mu = (-\alpha, 0, 0, 0)$, so

$$\nabla_i n_j = \alpha\, {}^{(4)}\Gamma^0{}_{ij} = \alpha\left({}^{(4)}g^{00}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0\ell}\, {}^{(4)}\Gamma_{\ell ij}\right)$$

$$\Gamma_{0ij} = \tfrac{1}{2}\left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - {}^{(4)}g_{ij,0}\right)$$

Thus

$$K_{ij} = -\frac{1}{2\alpha}\frac{\partial\, {}^{(3)}g_{ij}}{\partial t} + \ldots$$

So we can view the ex. curv. as the "velocity" of the 3-metric ${}^{(3)}g_{ij}$ (CONJUGATE MOMENTUM)

PHY 3E7XI THE 3+1 EQUATIONS ①

- HAVING DERIVED THE 3+1 EQNS IN COORDINATE-FREE FORM

(E1)
$$\mathcal{L}_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab}$$

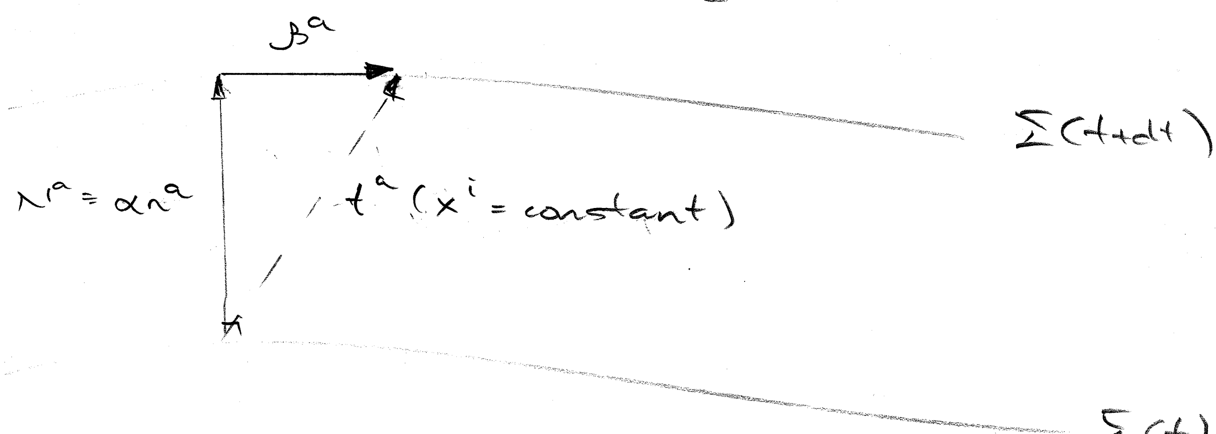
$$= -2\alpha \gamma_{ac} K^c{}_b + \mathcal{L}_\beta \gamma_{ab}$$

(E2)
$$\mathcal{L}_t K^a{}_b = \mathcal{L}_\beta K^a{}_b - D^a D_b \alpha$$

$$+ \alpha \left(R^a{}_b + K K^a{}_b + 8\pi \left(\tfrac{1}{2} \perp^a{}_b (S - \rho) - S^a{}_b \right) \right)$$

WE NOW WANT TO EXPRESS THE EQUATIONS AS TENSOR-COMPONENT EQUATIONS WITH RESPECT TO OUR 3+1 COORDINATE SYSTEM / COORDINATE BASIS

- RECALL THE BASIC 3+1 PICTURE



WHERE WE HAVE NOW LABELLED THE VARIOUS VECTORS USING COORDINATE-FREE NOTATION

- ALSO RECALL THAT IF $S_{\mu_1 \dots \mu_\ell}$ ARE THE COMPONENTS of A 4-TENSOR, THEN FOR ANY FOLIATION, $S_{i_1 \dots i_\ell}$ ARE THE COMPONENTS of A 3-TENSOR

• FINALLY, RECALL THE 3+1 DECOMPOSITION OF ${}^{(4)}g_{\mu\nu}$, n_μ, ${}^{(4)}g^{\mu\nu}$ & n^μ

$$ {}^{(4)}g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta^k\beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad n_\mu = (-\alpha, 0, 0, 0) $$

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix} \qquad n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right) $$

NOTE: $\beta_k = {}^{(3)}g_{kj}\beta^j \qquad {}^{(3)}g^{ik}\,{}^{(3)}g_{kj} = \delta^i{}_j$

CLAIM:
(1) $\gamma_{ij} = {}^{(3)}g_{ij}$
(2) $\gamma^{ij} = {}^{(3)}g^{ij} \qquad (\rightarrow \gamma^{ij}\gamma_{jk} = \delta^i{}_k)$
(3) $\perp^i{}_j = \delta^i{}_j$

$\Rightarrow$ GET VALID COMPONENT EQ^{NS} OF MOTION (IN $\{t, x^i\}$ COORDINATE BASIS) VIA $a \to i$, $b \to j$ ETC IN (E1), (E2) AND TREATING ALL QUANTITIES AS 3-TENSORS WHOSE INDICES ARE RAISED/LOWERED VIA 3-METRIC γ^{ij}, γ_{ij}

ALSO
$$ D_i V^j = \partial_i V^j + {}^{(3)}\Gamma^j{}_{ik} V^k $$

$$ D_i V_j = \partial_i V_j - {}^{(3)}\Gamma^k{}_{ij} V_k \qquad \text{etc} $$

WHERE
$$ {}^{(3)}\Gamma^i{}_{jk} = \gamma^{i\ell}\,{}^{(3)}\Gamma_{\ell jk} $$

$$ {}^{(3)}\Gamma_{ijk} = \frac{1}{2}\left(\partial_k\gamma_{ij} + \partial_j\gamma_{ik} - \partial_i\gamma_{jk}\right) $$

PROOF of CLAIM: USE $\gamma_{\mu\nu} = {}^{(4)}g_{\mu\nu} + n_\mu n_\nu$ AND ABOVE-QUOTED EXPRESSIONS FOR ${}^{(4)}g_{\mu\nu}, n_\mu$ ETC.

$$\gamma_{\mu\nu} = \begin{bmatrix} {}^{(4)}g_{00} + n_0 n_0 & {}^{(4)}g_{0j} + n_0 n_j \\ {}^{(4)}g_{0i} + n_0 n_i & {}^{(4)}g_{ij} + n_i n_j \end{bmatrix}$$

$$= \begin{bmatrix} -\alpha^2 + \beta^k \beta_k + \alpha^2 & \beta_j \\ \beta_i & {}^{(4)}g_{ij} \end{bmatrix}$$

$$= \begin{bmatrix} \beta^k \beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix}$$

$$\gamma^{\mu\nu} = \begin{bmatrix} {}^{(4)}g^{00} + n^0 n^0 & {}^{(4)}g^{0j} + n^0 n^j \\ {}^{(4)}g^{0i} + n^0 n^i & {}^{(4)}g^{ij} + n^i n^j \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{\alpha^2} + \frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} - \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} - \frac{\beta^j}{\alpha^2} & {}^{(3)}g^{ij} - \frac{\beta^i \beta^j}{\alpha^2} + \frac{\beta^i \beta^j}{\alpha^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & {}^{(3)}g^{ij} \end{bmatrix}$$

→ THUS $\gamma_{ij} = {}^{(3)}g_{ij}$; $\gamma^{ij} = {}^{(3)}g^{ij}$ AS CLAIMED (SO γ_{ij}, γ^{ij} ARE INVERSES)

ALSO: $\perp^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$

$$\perp^i{}_j = \delta^i{}_j + n^i n_j = \delta^i{}_j$$

LIE DERIVATIVES

- RECALL FROM EARLY DISCUSSION OF LIE DERIVATIVE (WALD C.2), AN "EXPANSION" OF $\mathcal{L}_V S^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$

$$\mathcal{L}_V S^{a_1\ldots a_k}{}_{b_1\ldots b_l} = V^c\left(\nabla_c S^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right) - \sum_{i=1}^{k} (\nabla_c V^{a_i}) S^{a_1 \ldots c \ldots a_k}{}_{b_1 \ldots b_l} + \sum_{j=1}^{l} (\nabla_{b_j} V^c) S^{a_1 \ldots a_k}{}_{b_1 \ldots c \ldots b_l}$$

THE ∇_a CAN BE <u>ANY</u> DERIVATIVE OPERATOR (NOT JUST THE METRIC COMPATIBLE ONE) INCLUDING THE ORDINARY DERIVATIVE ∂_a. (NOTE THAT WE HAVE LIE DERIVATIVE TERMS IN BOTH (E1) AND (E2))

CONVERTING (E1) TO 3+1 COMPONENT FORM

(1) $a \to i$, $b \to j$ etc

$$\mathcal{L}_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}$$

$$= -2\alpha \gamma_{ik} K^k{}_j + \mathcal{L}_\beta \gamma_{ij}$$

(2) CONVERT LIE DERIVATIVES TO EXPRESSIONS INVOLVING ORDINARY DERIVATIVES $_{,k} \equiv \frac{\partial}{\partial x^k} \equiv \partial_k$

$$\mathcal{L}_t (\ldots) = \frac{\partial}{\partial t}(\ldots) = \partial_t(\ldots)$$

$$\mathcal{L}_\beta \gamma_{ij} = \beta^k \partial_k \gamma_{ij} + \gamma_{ik}\partial_j \beta^k + \gamma_{kj} \partial_i \beta^k$$

(E1') $$\boxed{\partial_t \gamma_{ij} = -2\alpha \gamma_{ik} K^k{}_j + \beta^k \partial_k \gamma_{ij} + \gamma_{ik} \partial_j \beta^k + \gamma_{kj} \partial_i \beta^k}$$

- CAN REWRITE THIS USING D_i (γ_{ij}-COMPATIBLE DERIVATIVES) AS FOLLOWS

EXERCISE

$$\beta^k \partial_k \gamma_{ij} + \gamma_{ik} \partial_j \beta^k + \gamma_{kj} \partial_i \beta^k$$

$$= \beta^k \partial_k \gamma_{ij} + \partial_j (\gamma_{ik} \beta^k) - \beta^k \partial_j \gamma_{ik}$$

$$+ \partial_i (\gamma_{kj} \beta^k) - \beta^k \partial_i \gamma_{kj}$$

$$= \partial_i \beta_j + \partial_j \beta_i - (\partial_j \gamma_{ik} + \partial_i \gamma_{jk} - \partial_k \gamma_{ij}) \beta^k$$

$$= \partial_i \beta_j + \partial_j \beta_i - 2\, {}^{(3)}\Gamma_{kij} \beta^k$$

$$= \partial_i \beta_j + \partial_j \beta_i - 2\, {}^{(3)}\Gamma^k{}_{ij} \beta_k$$

$$= D_i \beta_j + D_j \beta_i$$

(E1'') $$\boxed{\partial_t \gamma_{ij} = -2\alpha \gamma_{ik} K^k{}_j + D_i \beta_j + D_j \beta_i}$$

(MTW 21.67)

EVOLUTION EQUATION FOR EXTRINSIC CURVATURE

$$\mathcal{L}_t K^a{}_b = \mathcal{L}_\beta K^a{}_b - D^a D_b \alpha$$

$$+ \alpha \left(R^a{}_b + K K^a{}_b + 8\pi \left(\tfrac{1}{2} \perp^a{}_b (S - \rho) - S^a{}_b \right) \right)$$

(1) AGAIN, $\mathcal{L}_t K^a{}_b \to \mathcal{L}_t K^i{}_j = \partial_t K^i{}_j$

(2) $\mathcal{L}_\beta K^a{}_b \to \mathcal{L}_\beta K^i{}_j = \beta^k \partial_k K^i{}_j - \partial_k \beta^i K^k{}_j + \partial_j \beta^k K^i{}_k$

$$\text{(E2')} \quad \boxed{\begin{aligned} \partial_t K^i{}_j &= \beta^k \partial_k K^i{}_j - \partial_k \beta^i K^k{}_j + \partial_j \beta^k K^i{}_k - D^i D_j \alpha \\ &+ \alpha \left(R^i{}_j + K K^i{}_j + 8\pi \left(\tfrac{1}{2} \delta^i{}_j (S - \rho) - S^i{}_j \right) \right) \end{aligned}}$$

WHERE: $D^i D_j \alpha = \gamma^{ik} D_k D_j \alpha$

$= \gamma^{ik} D_k (\partial_j \alpha)$

$= \gamma^{ik} (\partial_k \partial_j \alpha - \Gamma^l{}_{kj} \partial_l \alpha)$

$R^i{}_j = \gamma^{ik} R_{kj} = \gamma^{ik} R_{klj}{}^l$

$R_{ijk}{}^l = -2 \partial_{[i} \Gamma^l_{j]k} + 2 \Gamma^m{}_{k[i} \Gamma^l_{j]m}$

$\Gamma^i{}_{jk} \equiv {}^{(3)}\Gamma^i{}_{jk} = \frac{1}{2} \gamma^{il} (\partial_k \gamma_{lj} + \partial_j \gamma_{lk} - \partial_l \gamma_{jk})$

$K = K^i{}_i = \gamma^{ij} K_{ij}$

$S_{ij} = T_{ij} \quad ; \quad S^i{}_j = \gamma^{ik} S_{kj}$

$\rho = \alpha^2 T_{00}$