PHYS 410: Computational Physics Fall 2022

FINAL EXAM

Monday, December 12, 8:30–11:00 AM, IBLC 182

Please do not read past this page until you are instructed to begin the exam.

Please note the following:

1. This exam is completely closed book and electronics free: no books, notes, calculators, computers or cell phones are allowed.
2. You have 2 1/2 hours to complete the test.
3. The exam has three problems. Problems 1 and 2 are worth 10 points each, Problem 3 is worth 15, so there are a total of 35 points available. Marks for individual subproblems are given at the beginning of each subproblem.
4. Answer all questions in the exam booklets provided. Ensure that your name and student number are legibly entered on the front of each booklet that you use. Don't worry about the course section number.
5. There is not necessarily a tight correlation between the number of marks available for a subproblem and the nominal difficulty of the question. *In particular, you may find subproblem 3.4 significantly more challenging and time consuming than the other 4 and 5 point questions.*

Problem 1: [10 pts]

Problem 1.1: Derivative of a polynomial interpolant [5 pts]

Consider 3 equispaced data points:

$$(-h, f_{-1}),\ (0, f_0),\ (h, f_1)\,. \tag{1}$$

The standard representation of the Lagrange interpolating formula is

$$p(x) = \sum_{j=1}^{n} f_j l_j(x)\,,$$

where the maximum degree of the interpolating polynomial is $n-1$, the polynomials $l_j(x)$ satisfy

$$l_j(x_i) = \delta_{ji}$$

and δ_{ji} is the Kronecker delta function.

Using the standard representation, construct the Lagrange interpolating polynomial passing through the values (1), then evaluate the derivative of that polynomial at $x = 0$.

Problem 1.2: Richardson extrapolating an $O(h^2)$ FDA [5 pts]

Let x_j be a uniform finite difference mesh, such that $x_{j+1} - x_j = \Delta x$ for all j, and let $u_j \equiv u(x_j)$. Then the standard centred $O(\Delta x^2)$ approximation to the second derivative of $u(x)$ on the mesh is given by

$$\frac{d^2u}{dx^2}(x_j) \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}\,. \tag{2}$$

This approximation satisfies

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} = \frac{d^2u}{dx^2}(x_j) + \frac{1}{12}\Delta x^2 \frac{d^4u}{dx^4}(x_j) + O(\Delta x^4)\,. \tag{3}$$

Using (3) and Richardson extrapolation, construct an $O(h^4)$ approximation of the second derivative of $u(x)$ that uses the grid function values u_{j+2}, u_{j+1}, u_j, u_{j-1} and u_{j-2}.

Problem 2: [10 pts]

Consider the following ODE for the function $x \equiv x(t)$

$$\frac{d^2x}{dt^2} + \left(x\frac{dx}{dt}\right)^3 = s(t)\,, \tag{4}$$

where $s(t)$ is some specified source function. Equation (4) is to be solved with initial conditions

$$x(0) = x_0\,,$$

$$\frac{dx}{dt}(0) = v_0\,.$$

Note that the ODE is *nonlinear* in x and dx/dt.

Problem 2.1: Finite difference approximation of the ODE [2 pts]

Equation (4) is to be solved on a uniform finite difference mesh t^n such that $t^{n+1} - t^n = \Delta t$ for all n. The standard centred $O(\Delta t^2)$ approximation to the first derivative of $x(t)$ is given by

$$\frac{dx}{dt}(t^n) \approx \frac{x^{n+1} - x^{n-1}}{2\Delta t}\,. \tag{5}$$

Use (5) and (2) (the latter interpreted appropriately as the second time derivative) to write an $O(\Delta t^2)$ FDA to (4).

Problem 2.2: Initialization of the ODE [4 pts]

To initialize the FDA you wrote down, values for $x^1 \equiv x(0)$ and $x^2 \equiv x(\Delta t)$ must be computed up to and including terms of $O(\Delta t^2)$, so that the overall scheme is $O(\Delta t^2)$.

Specify appropriate values of x^1 and x^2 in terms of x_0, v_0, $s(0)$ and Δt.

Problem 2.3: Determining x^{n+1} [4 pts]

Describe in detail how you would use the FDA that you wrote down to determine x^{n+1} from x^n, x^{n-1} and other relevant quantities.

Problem 3: [15 pts]

Consider the following PDE for the function $u \equiv u(x,t)$:

$$u_t = u_x + \alpha u \,. \tag{6}$$

Here α is a real-valued constant. This equation is to be solved on the domain

$$0 \le x \le 1 \,, \qquad t \ge 0 \,, \tag{7}$$

with periodic boundary conditions, so that $x = 0$ and $x = 1$ are identified. The initial conditions are

$$u(x,0) = u_0(x) \,, \tag{8}$$

where $u_0(x)$ is a given function.

Equations (6) and (8) are to be solved on a finite difference mesh which is uniform in both x and t, so that we have $x_{j+1} - x_j = \Delta x$ for all j and $t^{n+1} - t^n = \Delta t$ for all n.

Problem 3.1: FDA of the PDE [2 pts]

Using the standard, centred, second order FDA of the first derivative, (5) (interpreted as a time derivative as appropriate) write a second order FDA of (6). (Adopt the usual finite difference notation $u_j^n \equiv u(x_j, t^n)$.)

Problem 3.2: Truncation Error of the FDA [4 pts]

Using Taylor series expansion about the point (x_j, t^n), determine the truncation error of the FDA. Your answer must include the full leading order terms, which will contain Δt and Δx raised to some power(s).

Problem 3.3: Initialization of the FDA [4 pts]

The finite difference scheme you wrote down for Problem 3.1 is initialized by specifying u_j^1 and u_j^2 for all j.

Describe fully how you would initialize u_j^1 and u_j^2 so that the overall finite difference scheme will be $O(\Delta t^2, \Delta x^2)$ accurate. Your answer should be given in terms of Δt, u_0 and derivatives of u_0.

Problem 3.4: Stability Analysis of the FDA [5 pts]

Your final task is to perform a von Neumann stability analysis of the scheme you wrote down for Problem 3.1. You can make the usual assumptions, which in this case amounts to assuming that we are solving the problem on the unbounded domain $-\infty < x < \infty$. You may find the following formulae useful:

Fourier transform

$$\tilde{\mathbf{u}}^n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx}\, \mathbf{u}^n(x)\, dx \,.$$

Inverse Fourier transform

$$\mathbf{u}^n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikx}\, \tilde{\mathbf{u}}^n(k)\, dk \,.$$

Perform a von Neumann analysis of the finite difference scheme you wrote down for Problem 3.1. Your analysis should result in a condition on $\lambda \equiv \Delta t/\Delta x$ for stability of the scheme.