

PHYSICS 410

THE RUNGE-KUTTA-FEHLBERG METHOD

- Uses two RK methods, one $O(h^4)$, one $O(h^5)$
- As noted previously, for a given order there are in general an infinite number of possible RK methods
- Clever approach due to Fehlberg is to choose methods so that they use identical intermediate function evaluations f_0, f_1, f_2, f_3, f_4 and, for the $O(h^5)$ method, f_5

$$f_0 = f(x_0, y_0)$$

$$f_1 = f\left(x_0 + \frac{h}{4}, y_0 + \frac{h}{4}f_0\right)$$

$$f_2 = f\left(x_0 + \frac{3h}{8}, y_0 + \frac{3h}{32}f_0 + \frac{9h}{32}f_1\right)$$

$$f_3 = f\left(x_0 + \frac{12h}{13}, y_0 + \frac{1932h}{2197}f_0 - \frac{7200h}{2197}f_1 + \frac{7296h}{2197}f_2\right)$$

$$f_4 = f\left(x_0 + h, y_0 + \frac{439h}{216}f_0 - 8hf_1 + \frac{3680}{513}f_2 - \frac{845h}{4104}f_3\right)$$

$$f_5 = f\left(x_0 + \frac{h}{2}, y_0 - \frac{8h}{27}f_0 + 2hf_1 - \frac{3544h}{2565}f_2 + \frac{1859h}{4104}f_3 - \frac{11h}{40}f_4\right)$$

- With these definitions, the $O(h^4)$ approximation is

$$y = y_0 + h \left(\frac{25}{216} f_0 + \frac{1408}{2565} f_2 + \frac{2197}{4104} f_3 - \frac{1}{5} f_4 \right)$$

and the $O(h^5)$ approximation is

$$\tilde{y} = y_0 + h \left(\frac{16}{135} f_0 + \frac{6656}{12825} f_2 + \frac{28561}{56430} f_3 - \frac{9}{50} f_4 + \frac{2}{55} f_5 \right)$$

- Can compute the approximate error directly

$$e_{\text{est}} = \tilde{y} - y = h \left(\frac{1}{360} f_0 - \frac{128}{4275} f_2 - \frac{2197}{75240} f_3 + \frac{1}{50} f_4 + \frac{2}{55} f_5 \right)$$