PHY329 WAVE EQUATION (note correction) OCT 23, 1996

GENERATING INITIAL DATA: $\hat{u}^0_j$ and $\hat{u}^1_j$

First note a slight change in notation from the original lecture notes. Here, for ease of exposition, $\hat{u}^n_j$ is the quantity which satisfies the difference equation (22.4), while u^n_j satisfies the differential equation (22.2).

We begin by assuming that our difference solution will admit a Richardson expansion

$$\hat{u}(x,t) = u(x,t) + h^2 e_2(x,t) + h^4 e_4(x,t) + \cdots \qquad (22.7)$$

and ask what this implies about how accurately we must specify $\hat{u}^0_j$ and $\hat{u}^1_j$. First recall that the problem prescription includes the initial conditions:

$$u\,(x,0) = f\,(x) \qquad\qquad u_t\,(x,0) = g\,(x)$$

Clearly then, we can (and might as well) specify $\hat{u}^0_j$ exactly

$$\hat{u}^0_j = f_j$$

which implies that $e_2(x,0) = 0$. Then in order for (22.7) to hold, we must have

$$\begin{aligned}\hat{u}^1_j &\equiv \hat{u}(x_j,\,\triangle t\,)\\ &= u(x_j,\,\triangle t\,) + h^2 e_2(x_j,\,\triangle t\,) + O(h^4)\end{aligned}$$

which implies that

$$\hat{u}(x_j,\,\triangle t\,) - u(x_j,\,\triangle t\,) = h^2 e_2(x_j,\,\triangle t\,) + O(h^4)$$

But

$$\begin{aligned}e_2(x_j,\,\triangle t\,) &= e_2(x_j,0) + \,\triangle t\,\frac{\partial e_2}{\partial t}(x_j,0) \,+\, O(\,\triangle t^{\,2})\\ &= \,\triangle t\,\frac{\partial e_2}{\partial t}(x_j,0) + O(\,\triangle t^{\,2}) = \lambda h \frac{\partial e_2}{\partial t}(x_j,0) + O(\,\triangle t^{\,2})\end{aligned}$$

Thus,

$$\hat{u}(x_j,\,\triangle t\,) - u(x_j,\,\triangle t\,) = h^2\left(\lambda h \frac{\partial e_2}{\partial t}(x_j,0)\right) \,+\, \cdots = O(h^3)$$

So we see that we must specify $\hat{u}^1_j$ to $O(h^3)$ accuracy, *not* $O(h^2)$ as was erroneously claimed in the lecture. However, this is readily done by simply Taylor-expanding to higher order:

$$u(x_j,\,\triangle t\,) = u(x_j,0) + \,\triangle t\,u_t(x_j,0) + \frac{1}{2}\triangle t^{\,2}\,u_{tt}(x_j,0) + O(\,\triangle t^{\,3})$$

Recalling (22.6) where we specified the initial conditions in terms of initially left-moving and right-moving profiles, $l(x)$ and $r(x)$, and their derivatives, $l'(x)$ and $r'(x)$:

$$\begin{aligned}u\,(x,0) &= l\,(x) + r\,(x)\\ u_t\,(x,0) &= l'\,(x) - r'\,(x)\end{aligned}$$

and noting that

$$u_{tt}\,(x,0) = l''\,(x) + r''\,(x)$$

we have for our initialization:

$$\hat{u}^0_j = l_j + r_j \qquad (22.7)$$

$$\hat{u}^1_j = l_j + r_j + \,\triangle t\,\left(l'_j - r'_j\right) + \frac{1}{2}\triangle t^{\,2}\left(l''_j + r''_j\right) \qquad (22.8)$$