FINITE DIFFERENCE SOLN OF PARTIAL DIFFERENTIAL EQUATIONS PHY 381C 3/18/97

OVERVIEW

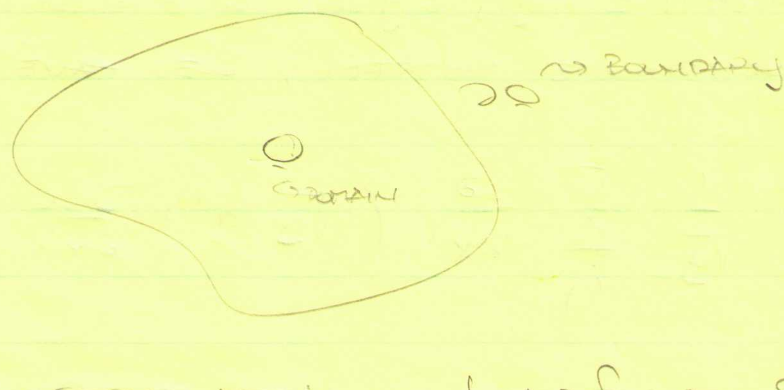
(A) • TIME-INDEPENDENT (ELLIPTIC) EQUATIONS

(B) • TIME-DEPENDENT (PARABOLIC, HYPERBOLIC, NON-LINEAR / DISPERSIVE, SCHRODINGER) EQUATIONS

(A) TIME-INDEPENDENT (ELLIPTIC) EQUATIONS

- FOR OUR PURPOSES "ELLIPTIC EQUATION" WILL BE SYNONYMOUS WITH "BOUNDARY VALUE PROBLEM" AND ALTHOUGH ELLIPTIC PROBS IN 3D ARE NOW ROUTINELY SOLVED, THE DEVELOPMENT WILL FOCUS ON 2D MODEL PROBLEMS

B.V.P



ABSTRACT FORMULATION $Lu = f$ on Ω

$Bu = g$ on $\partial\Omega$

BOUNDARY CONDITIONS NEED TO BE SPECIFIED EVERYWHERE ON $\partial\Omega$

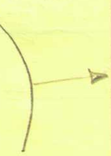
APPLICATIONS : ALL SORTS OF STEADY-STATE / ACTION-AT-A-DISTANCE PROBLEMS (UBIQUITOUS) EXERCISE : FIND

TYPES of BOUNDARY CONDITIONS

(i) DIRICHLET: $u\big|_{\partial\Omega}$ GIVEN

(ii) NEUMANN: $\dfrac{\partial u}{\partial n}\Big|_{\partial\Omega}$ GIVEN

↳ NORMAL DERIVATIVE



(iii) MIXED (ROBBIN): $\alpha u + \beta \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega}$ GIVEN

MODEL PROBLEM (2-D, "CARTESIAN" COORDINATES x, y)

ASIDE: ARGUABLY THE MOST PHYSICALLY IMPORTANT ELLIPTIC OPERATOR (ON SCALAR FUNCTIONS) IS THE COVARIANT LAPLACIAN $\Delta u(x^i)$ WHERE $x^i \equiv \{x^1, x^2, x^3\}$ IS ANY THREE-DIMENSIONAL COORDINATE SYSTEM. GIVEN THE (FLAT) METRIC $g_{ij} = g_{ji}$ (6 INDEPENDENT COMPONENTS) IN THAT COORDINATE SYSTEM, WE HAVE

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial u}{\partial x^j}\right)$$

WHERE g IS THE DETERMINANT OF g_{ij} VIEWED AS A 3x3 MATRIX ($\sqrt{g}$ = root of this det) AND g^{ij} IS THE INVERSE METRIC

$$\sum_k g^{ik} g_{kj} = \delta^i{}_j$$

↳ KRONECKER DELTA

ELLIPTIC PDE'S PHY 381C

MODEL PROBLEM

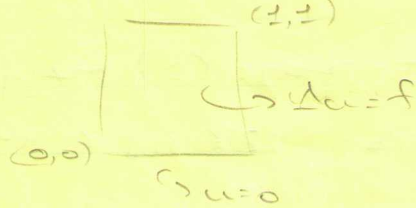
$$\Delta u(x,y) = u_{xx} + u_{yy} = f(x,y) \qquad (1a)$$

(COVARIANT) LAPLACIAN OPERATOR

ON $[0,1] \times [0,1]$ WITH BOUNDARY CONDITIONS

$$u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0 \qquad (1b)$$

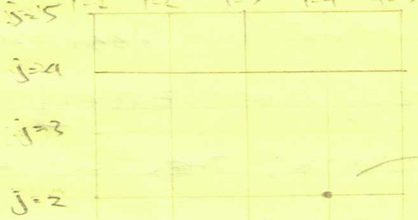
(HOMOGENEOUS DIRICHLET B.C.'S)



DISCRETIZATION OF MODEL PROBLEM

(1) $\Omega \to \Omega^h$ UNIFORM DISCRETIZATION, SINGLE MESH SCALE $h = 1/(n-1)$

$n \equiv$ # OF GRID POINTS PER EDGE, TOTAL # OF GRID POINTS n^2



DISCRETE DOMAIN (Ω^h)

$$\{(x_i, y_j) \mid x_i = x_0 + (i-1)h,\; y_j = y_0 + (j-1)h\}$$

$x_0 = 0$, $y_0 = 0$ IN OUR EXAMPLE

(2) FINITE DIFFERENCE APPROXIMATION $\begin{matrix} Lu = f \\ Bu = g \end{matrix} \to \begin{matrix} L^h u^h = f^h \\ B^h u^h = g^h \end{matrix}$

– "NAIVE" APPROACH BASED ON 1-D EXPERIENCE WORKS

$$u_{xx} \rightarrow h^{-2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) \qquad \left(O(h^2)\ \text{t.e.}\right)$$

$$u_{yy} \rightarrow h^{-2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) \qquad \left(O(h^2)\ \text{t.e.}\right)$$

$$L^h u^h = f^h$$

$$\therefore \boxed{u_{xx} + u_{yy} = h^{-2}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right) + O(h^2)} \qquad (2)$$

(SHOULD VERIFY USING TAYLOR SERIES, JUST TO KEEP IN PRACTICE) STENCIL

$$h^{-2} \begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}$$

"USUAL 5-PT STENCIL"

NOTE THAT (2) MAY BE APPLIED AT ALL SO-CALLED INTERIOR POINTS $2 \le i,j \le n-1$ $(n-2)^2$ IN TOTAL

BOUNDARY CONDITIONS PROVIDE (TRIVIAL EQUATIONS) FOR

$u_{1,2} \cdots u_{1,n-1}$	$n-2$
$u_{n,2} \cdots u_{n,n-1}$	$n-2$
$u_{2,1} \cdots u_{n-1,1}$	$n-2$
$u_{2,n} \cdots u_{n-1,n}$	$n-2$

TOTAL # OF EQUATIONS/UNKNOWNS $(n-2)^2 + 4(n-2)$

NOTE: COULD ALSO VIEW 4 "CORNER VALUES" $u_{11}, u_{1n}, u_{n1}, u_{nn}$ AS BEING SPECIFIED (0) EVEN THOUGH THEY COMPLETELY "DECOUPLE" FROM REMAINDER OF EQUATIONS; THEN WE HAVE n^2 EQUATIONS IN n^2 UNKNOWNS – SUCH DECISIONS MAY MAKE BOOKKEEPING (I.E. PROGRAMMING) EASIER

ANOTHER APPROACH: APPLY (2) VERBATIM ONLY FOR $3 \le i,j \le n-2$

FOR NEXT-TO-EXTREMAL ROWS/COLUMNS USE (2) BUT WITH DIRICHLET CONDITIONS DIRECTLY INCORPORATED

EXAMPLE: $j=2$, $i=2,\ldots,n-1$ BC $u_{j+1,i}=0$

$\rightarrow h^{-2}(u_{i+1,j}+u_{i-1,j}+u_{j-1,i}-4u_{i,j})=f_{i,j}$

POINT IS THAT, EVEN FOR SIMPLE DIFFERENCE SCHEMES FOR SIMPLE BVP'S, CONSIDERABLE FLEXIBILITY IN PRECISE DETAILS OF DISCRETIZATION; NO HARD, FAST RULES, EXCEPT POSSIBLY THAT CLARITY/STRAIGHTFORWARDNESS SHOULD BE PREFERRED.

SOLUTION OF DISCRETE EQUATIONS $L^h u^h = f^h$ $B^h u^h = g^h$

- HUGE LITERATURE ON THIS SUBJECT; MUCH PROGRESS DEVELOPMENT IN 70'S AND 80'S
- IF YOU NEED AN ELLIPTIC SOLVER IN A SERIOUS WAY (PARTICULARLY IF THE EQN IS STANDALONE); WELL ADVISED TO USE PACKAGE - ELLPACK, PLT2D, PLT3D
- WILL ESCHEW "BLACK BOXES" FOR THE TIME BEING SINCE HANDS-ON EXPERIENCE HERE IS VALUABLE
- NOTE THAT DISCRETIZATION OF (1) USING (2) RESULTS IN A LARGE SPARSE LINEAR SYSTEM OF EQUATIONS

SPECIFICALLY, CONSIDER SOME NUMBERING (LINEARIZATION) OF THE UNKNOWNS $u_{i,j} \rightarrow u_k$ SO THAT WE CAN AGAIN VIEW THE DISCRETE GRID FUNCTION AS

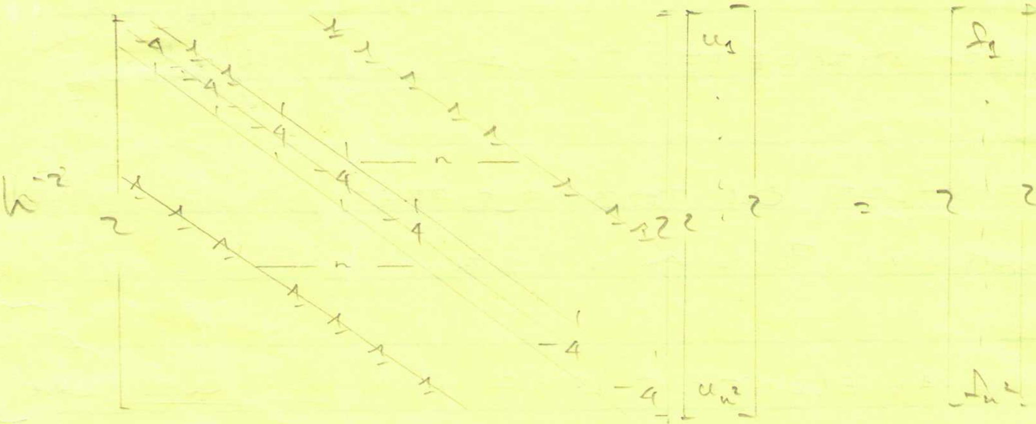
A SINGLE VECTOR $\underset{\sim}{u}$

EXAMPLE:

$$i, j \rightarrow k = (j-1)n + i$$
$$i+1, j \rightarrow k+1$$
$$i-1, j \rightarrow k-1$$
$$i, j+1 \rightarrow k+n$$
$$i, j-1 \rightarrow k-n$$

$$(2) \rightarrow \boxed{h^{-2}\left(u_{k+1} + u_{k-1} + u_{k+n} + u_{k-n} - 4u_k\right) - f_k = 0}$$

SCHEMATICALLY, IN MATRIX FORM WE HAVE



NB: BOUNDARY CONDITIONS NOT PROPERLY INCORPORATED

THUS, TO DETERMINE OUR DISCRETE $\underset{\sim}{u} = \underset{\sim}{u}^h$ WE NEED TO SOLVE THE LINEAR SYSTEM

$$\underset{\approx}{L}\,\underset{\sim}{u} = \underset{\sim}{f}$$

WHERE $\underset{\approx}{L}$ IS A (ROUGHLY) $n^2 \times n^2$ SPARSE, BANDED MATRIX WITH BANDWIDTH $2n+1$

→ BAD NEWS!

FOR SQUARE MATRIX OF DIMENSION N, BANDWIDTH w, TIME TO SOLVE $\underset{\approx}{L}\,\underset{\sim}{u} = \underset{\sim}{f}$ IS $O(C_w N)$ WHERE C_w IS A

CONSTANT.

NOTE: FOR FDAs SUCH AS WE'VE PREVIOUSLY STUDIED AND ARE CURRENTLY CONSIDERING, DIMENSION of MATRIX ~ # OF GRID POINTS ~ # OF DISCRETE DEGREES of FREEDOM. BEST WE SHOULD EXPECT FROM APPROX SOLN IS COMPUTATIONAL WORK / TIME

$$W = O(N)$$

BUT ALSO IS SCALING BEHAVIOUR WE SHOULD AIM FOR (BRANDT'S "GOLDEN RULE OF NUMERICAL ANALYSIS" - COMPUTATIONAL EFFORT SHOULD SCALE LINEARLY WITH "PHYSICAL PROCESS")

FOR 1-D SYSTEMS, $\underset{\text{\# OF DISCRETE D.O.F.}}{N} \sim \overset{\text{\# OF GRID POINTS PER DIMENSION}}{n}$, AND TRI-DIAGONAL, PENTA-DIAGONAL SYSTEMS, ETC. CAN BE SOLVED IN $O(N)$ OPS (RECALL: INTERESTED IN LIMIT $h \to 0$, $n, N \to \infty$)

FOR OUR 2-D PROBLEMS, BANDWIDTH OF $\underset{\approx}{A}$ IS NOT CONSTANT AS $h \to 0$ $(2n+1 \sim 2h^{-1}+1)$ SO DIRECT SOLUTION BECOMES MORE EXPENSIVE PER GRID POINT AS $h \to 0$

CONSEQUENTLY, ITERATIVE METHODS SUCH AS RELAXATION AND MORE RECENTLY, MULTI-GRID HAVE COME TO DOMINATE SOL^{N} OF LINEAR SYSTEMS ARISING IN ELLIPTIC PDEs., WILL FOCUS ON THESE

RELAXATION (ITERATIVE SOL'N OF SPARSE LINEAR SYSTEMS)

- ADVANTAGES: EASY (ALMOST TRIVIAL) TO CODE
  WORKS FOR MANY DISCRETE BVPs
  EXTREMELY STORAGE-EFFICIENT

- DISADVANTAGES: SLOW CONVERGENCE (TIME PER GRID POINT GOES UP AS $h \to 0$)
  DOESN'T WORK FOR ALL DISCRETE BVPs

SEVERE ENOUGH THAT RELAXATION SHOULD ALMOST NEVER BE USED IN PRACTICE (EXCEPT ON MESHES WITH RELATIVELY FEW, SAY 32 OR LESS, PTS PER EDGE)

WHY STUDY?

- FORMS CORE OF BEST GENERAL METHOD FOR SOLVING ELLIPTIC EQUATIONS (MULTI-GRID)
- GOOD TEST 2D APPLICATION, GET YOU WORKING WITH 2D ARRAYS, 2D-VISUALIZATION ETC.

GAUSS-SEIDEL (AND JACOBI) RELAXATION AND SINGLE UNKNOWN

BASIC IDEA. ASSOCIATE SINGLE EQUATION WITH EACH MESH POINT IN Ω^h. "REPEATEDLY" "SWEEP" THROUGH THE MESH, VISITING EACH MESH POINT IN SOME ORDER. EACH TIME A POINT IS VISITED, ADJUST THE VALUE OF THE UNKNOWN SO THAT THE CORRESPONDING

ADOPT "RESIDUAL-BASED" APPROACH TO LOCALLY SATISFYING EQUATIONS.

NOTATION: (TEMPORARILY "FORGET" ABOUT B.C.'S)

$$L^h u^h = f^h \rightarrow F^h[u^h] = 0$$

↳ "EXACT" SOLN of DIFFERENCE EQUATIONS

ITERATIVE TECHNIQUE $\tilde{u}^h$

↳ CURRENT ("WORKING")

APPROXIMATION TO u^h $\quad \lim_{n \to \infty} \tilde{u}^h = u^h$

↳ ITERATION #

(GLOBAL) RESIDUAL: $\tilde{r}^h \equiv L^h \tilde{u}^h - f^h$

→ CONVENIENT TO USE MESH-BASED INDEXING

COMPUTING ij'TH COMPONENT OF $\tilde{r}^h$

$\tilde{r}_{ij} \equiv \tilde{r}^h_{ij}$

$$\tilde{r}_{ij} = \left[L^h \tilde{u}^h - f^h \right]_{ij} \equiv \left[F^h [u^h] \right]_{ij}$$

$$= h^{-2} \left(\tilde{u}_{i+1,j} + \tilde{u}_{i-1,j} + \tilde{u}_{i,j+1} + \tilde{u}_{i,j-1} - 4\tilde{u}_{i,j} \right) - f_{ij}$$

→ NOTE SWITCH IN SIGN CONVENTION

UPDATE $\quad \tilde{u}^{(n)}_{ij} \rightarrow \tilde{u}^{(n+1)}_{ij} = \tilde{u}^{(n)}_{ij} + \delta \tilde{u}^{(n)}_{ij}$

$$= \tilde{u}^{(n)}_{ij} - \frac{\tilde{r}_{ij}}{\left. \dfrac{\partial F^h_{ij}}{\partial u_{ij}} \right|_{u = \tilde{u}^{(n)}_{ij}}} \qquad \left. \frac{\partial F^h_{ij}}{\partial u_{ij}} \right|_{u = \tilde{u}^{(n)}_{ij}} \equiv J^D_{ij}$$

$$= \tilde{u}^{(n)}_{ij} - \frac{\tilde{r}^h_{ij}}{(-4h^{-2})} = \tilde{u}^{(n)}_{ij} + \frac{1}{4} h^2 \tilde{r}_{ij}$$

COMPUTATION OF RESIDUAL REQUIRES SOME CLARIFICATION. HAVE ITERATIVE METHOD $\tilde{u}^{(n)} \to \tilde{u}^{(n+1)}$ WHICH WORKS ON COMPONENT BY COMPONENT BASIS; COULD USE (a) "ALL-OLD" VALUES FROM ITERATION (n) OR (b) "SOME OLD, SOME NEW"

(a) JACOBI RELAXATION

$$\tilde{r}_{ij}^{(n)} = h^{-2}\left(\tilde{u}_{i+1,j}^{(n)} + \tilde{u}_{i-1,j}^{(n)} + \tilde{u}_{i,j+1}^{(n)} + \tilde{u}_{i,j-1}^{(n)} - 4\tilde{u}_{i,j}^{(n)}\right) - f_{ij}$$

$$\tilde{u}_{ij}^{(n+1)} = \tilde{u}_{ij}^{(n)} - \tilde{r}_{ij}^{(n)} / J_{ij}^{D} \quad \text{"DIAGONAL JACOBI ELEMENT"} \qquad (J)$$

(b) GAUSS-SEIDEL RELAXATION (ASSUME "LEXICOGRAPHIC" ORDERING OF UNKNOWNS $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$

$$\tilde{r}_{ij}^{(n)} = h^{-2}\left(\tilde{u}_{i+1,j}^{(n)} + \tilde{u}_{i-1,j}^{(n+1)} + \tilde{u}_{i,j+1}^{(n)} + \tilde{u}_{i,j-1}^{(n+1)} - 4\tilde{u}_{i,j}^{(n)}\right) - f_{ij}$$

$$\tilde{u}_{ij}^{(n+1)} = \tilde{u}_{ij}^{(n)} - \tilde{r}_{ij}^{(n)} / J_{ij}^{D} \qquad (GS)$$

NOTES

- ORDER WE "VISIT" UNKNOWNS $\tilde{u}_{ij}$ CLEARLY IRRELEVANT FOR JACOBI; HENCE ALSO KNOWN AS "SIMULTANEOUS" RELAXATION; PARALLELIZABLE; REQUIRES STORAGE FOR BOTH NEW & OLD $\tilde{u}_{ij}$ (2N)

- ONLY NEED STORAGE FOR CURRENT ESTIMATE $\tilde{u}_{ij}$ (N) SWEEPING ORDER DOES MATTER, LEXI- DOESN'T PARALLELIZE BUT RED-BLACK ORDERING DOES, PLUS HAS OTHER ADVANTAGES FOR MULTI-GRID (RBGS WILL BE OUR RELAXATION METHOD OF CHOICE)

RED-BLACK ORDERING (APPEAL TO RED / BLACK SQUARES ON CHESS BOARD)

VISIT ALL "RED" POINTS $\mathrm{mod}(i+j,2)=0$
" " " "BLACK" POINTS $\mathrm{mod}(i+j,2)=1$

WITHIN EACH SUBSET, UPDATE ORDER IS IRRELEVANT

CONVERGENCE

ROUGH GUIDE: GS WILL CONVERGE IF LINEAR SYSTEM IS DIAGONALLY DOMINANT

$$L^h u^h = f^h \rightarrow \underset{\sim}{A}\,\underset{\sim}{u} = \underset{\sim}{b} \qquad |a_{ii}| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} |a_{ij}| \,, \quad i=1,\ldots,n$$

WITH STRICT INEQUALITY HOLDING FOR AT LEAST ONE VALUE OF i

MONITORING CONVERGENCE

RESIDUAL NORM $\|\tilde{r}^h\| \rightarrow 0$ AS $(n)\rightarrow\infty$
UPDATE NORM $\|\tilde{u}^{(n+1)} - \tilde{u}^{(n)}\| \rightarrow 0$

IN PRACTICE, MONITORING RESIDUAL NORM IS STRAIGHTFORWARD AND SUFFICIENT (SEE DISCUSSION OF MULTI-GRID TO COME)

$$\rightarrow \tilde{r}^h = L^h \tilde{u}^h - f^h$$

WITH GAUSS SEIDEL, RESIDUAL IS CONSTANTLY CHANGING

DEFINE CONCEPT OF "RUNNING RESIDUAL", COLLECTION OF INDIVIDUAL RESIDUALS COMPUTED DURING SINGLE RELAXATION SWEEP.

IMPLEMENTATION

gs2d <level> [<order> <nsweep> <init>]

PERFORMS GAUSS-SEIDEL RELAXATION ON THE MODEL SYSTEM

TEST SOLUTION (USE SAME TECHNIQUE OF SPECIFYING $u(x,y)$ WHICH SATISFIES B.C.'S, COMPUTING CORRESPONDING $f(x,y) = u_{xx} + u_{yy}$

$$u(x,y) = \sin(l_x x) \sin(l_y y)$$

l_x, l_y INTEGERS

$$\rightarrow f(x,y) = -(l_x^2 + l_y^2) \, u(x,y)$$

$u_{exact}(x,y)$

CONVERGENCE RATE OF ITERATIVE PROCEDURE

$$\underset{\sim}{u}^{(0)} \to \underset{\sim}{u}^{(1)} \to \cdots \to \underset{\sim}{u}^{(n)} \to \underset{\sim}{u}^{(n+1)} \to \cdots \to \underset{\sim}{u}$$

$$\underset{\sim}{r}^{(0)} \to \underset{\sim}{r}^{(1)} \to \cdots \; \underset{\sim}{r}^{(n)} \to \underset{\sim}{r}^{(n+1)} \to \cdots \to \underset{\sim}{0}$$

$$\underset{\sim}{e}^{(0)} \to \underset{\sim}{e}^{(1)} \to \cdots \; \underset{\sim}{e}^{(n)} \to \underset{\sim}{e}^{(n+1)} \to \cdots \to \underset{\sim}{0}$$

WHERE $\underset{\sim}{e}^{(n)} \equiv \underset{\sim}{u} - \underset{\sim}{u}^{(n)}$. FOR LINEAR RELAXATION (AND BASIC IDEA EXTENDS TO NON-LINEAR CASE)

$$\boxed{\underset{\sim}{e}^{(n+1)} = G\,\underset{\sim}{e}^{(n)} = G^n\,\underset{\sim}{e}^{(0)}} \qquad (5)$$

$G \equiv$ "ERROR AMPLIFICATION MATRIX"
(WILL ALSO CONSIDER "RESIDUAL AMPLIFICATION MATRIX")

ASYMPTOTICALLY, CONVERGENCE OF ITERATION DETERMINED BY

SPECTRAL RADIUS of $G \equiv \rho(G)$

$$\rho(G) \equiv \max_i |\lambda_i(G)| \qquad \lambda_i \equiv \text{EIGENVALUES of } G$$

I.E., IN GENERAL

$$\boxed{\lim_{n\to\infty} \frac{\|\underset{\sim}{e}^{(n+1)}\|}{\|\underset{\sim}{e}^{(n)}\|} = \rho(G)} \qquad (6)$$

AND WE CAN THEN DEFINE ASYMPTOTIC CONVERGENCE RATE

$$\boxed{R \equiv \log_{10}(\rho^{-1})} \qquad (7)$$

$R^{-1} \equiv$ # OF ITERATIONS NECESSARY TO ASYMPTOTICALLY

COST OF SOLVING $L^h u^h = f^h$ VIA RELAXATION.

COST = COMPUTATIONAL WORK ~ CPU TIME (COMPLEXITY ANALYSIS)

ASSUME: (1) d-DIMENSIONAL DOMAIN Ω ($d = 1, 2, 3$. MOST COMMON)

(2) n GRID PTS PER EDGE of UNIFORM GRID Ω^h

→ (3) TOTAL # OF GRID POINTS (DISCRETE D.o.F.) $N = n^d$

RECALL: BEST CASE → $W = O(N)$

↳ COMPUTATIONAL WORK / COST

WHAT IS W?

- IDEALLY, TOTAL WORK TO SOLVE TO SOME PRESCRIBED TOLERANCE ($\|\tilde{u}^h - u^h\| \le \epsilon_u$ or $\|\tilde{r}^h\| \le \epsilon_r$) BUT THIS IS DIFFICULT TO QUANTIFY; DEPENDS ON INITIAL GUESS FOR $\tilde{u}^h$, ETC.
- WILL THUS DEFINE W TO BE WORK REQUIRED TO REDUCE $\|e^h\|$ BY AN ORDER of MAG. (CLEARLY THIS SHOULD SUFFICE FOR PURPOSE OF COMPARING METHODS)

CONVERGENCE RATE OF GAUSS-SEIDEL

- FOR OUR 2-D MODEL PROBLEM (AND FOR 1-D, 3-D VERSIONS)

$$\boxed{\rho(G_{GS}) = 1 - O(h^2)} \tag{8}$$

THUS,

$$\boxed{W_{GS} = O(h^{-2} \text{ SWEEPS}) = O(n^2 \text{ SWEEPS})}$$

EACH SWEEP COSTS $O(n^d) = O(N)$. THUS

$$\boxed{W_{GS} = O(n^2 N) = O(N^{2/d} N) = O(N^{(d+2)/d})} \tag{9}$$

d	W/GS
1	$O(N^3)$
2	$O(N^2)$
3	$O(N^{5/3})$

SCALING GETS BETTER AS d INCREASES, BUT $O(N^2)$, $O(N^{5/3})$ ARE STILL PRETTY BAD

SUCCESSIVE OVERRELAXATION (SOR)

$$\boxed{u_k^{(n+1)} = \omega \, \hat{u}_k^{(n+1)} + (1-\omega)\, u_k^{(n)}} \qquad (10)$$

"ij" FOR 2D MODEL PROBLEM

$\omega \equiv$ "OVERRELAXATION PARAMETER", TYPICALLY NEED $0 < \omega < 2$ FOR CONVERGENCE, "TRUE" OVERRELAXATION $\to$ $1 < \omega < 2$

$\hat{u}_k^{(n+1)} \equiv$ VALUE COMPUTED FROM NORMAL GAUSS-SEIDEL ITERATION

NOTE: $\omega = 1 \Rightarrow$ GAUSS-SEIDEL

- UNDER IDEAL CONDITIONS, REDUCES # OF SWEEPS REQUIRED TO $O(n)$, THUS

$$\boxed{W_{SOR} = O(n N) = O(N^{1/d} N) = O\left(N^{d+1/d}\right)} \qquad (11)$$

d	W/SOR
1	$O(N^2)$
2	$O(N^{3/2})$
3	$O(N^{4/3})$

$\to$ OPTIMAL SOR NOT UNREASONABLE FOR MODERATELY-SIZED 3-d MESHES

• EXCEPT FOR PROBLEMS WHERE $\rho_{GS} = \rho(G_{GS})$ CAN BE COMPUTED EXPLICITLY, ω_{OPT} WILL GENERALLY NEED TO BE DETERMINED <u>EMPIRICALLY</u> ON A CASE-BY-CASE <u>AND</u> RESOLUTION-BY-RESOLUTION BASIS. IF ρ_{GS} <u>IS</u> KNOWN

$$\boxed{\omega_{OPT} = \frac{2}{1+(1-\rho_{GS})^{\frac{1}{2}}}} \qquad (12)$$

<u>RELAXATION AS SMOOTHER (EXAMPLE) - PRELUDE TO MULTI-GRID</u>

• PROVIDES INSIGHT INTO WHAT RELAXATION <u>DOES</u> DO WELL (SMOOTH) AND (SHORTLY), WHY RELAXATION IS SO INTEGRAL TO MULTI-GRID

<u>1-d MODEL PROBLEM</u>

$$Lu = \frac{d^2u}{dx^2} = f(x) \qquad 0 \le x \le 1 \qquad x(0) = x(1) = 0 \qquad (13)$$

<u>DISCRETIZATION</u> : $\Omega^h = \{(j-1)h,\ j = 1 \ldots n;\ n = h^{-1}+1\}$

$$L^h u^h = f^h \rightarrow h^{-2}(u_{j+1} - 2u_j + u_{j-1}) = f_j \qquad j = 2, \ldots, n-1 \qquad (14)$$

$$u_1 = u_n = 0$$

CAN VIEW L^h AS $(n-2) \times (n-2)$ MTX: $h^{-2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}$

• CONSIDER THE FOLLOWING RELAXATION METHOD ("DAMPED JACOBI") CHOSEN FOR EASE OF ANALYSIS (STILL REPRESENTATIVE)

$$\boxed{\tilde{u}^{(n+1)} = \tilde{u}^{(n)} - \omega D^{-1} \tilde{r}^{(n)}} \qquad (15)$$

$D \equiv$ MAIN DIAGONAL OF L^h $\left(-2h^{-2}\mathbb{1}\right)$

(FOR $\omega = 1$ THIS IS JUST THE USUAL JACOBI SCHEME)

- CONSIDER THE EFFECT OF (15) ON $\tilde{r}$

$$\tilde{r}^{(n+1)} \equiv L^h \tilde{u}^{(n+1)} - f^h$$

$$= L^h\left(\tilde{u}^{(n)} - \omega D^{-1}\tilde{r}^{(n)}\right) - f^h$$

$$= \left(\mathbb{1} - \omega L^h D^{-1}\right)\tilde{r}^{n}$$

$$= G_R \tilde{r}^{(n)} \quad \text{WHERE} \quad G_R \equiv \mathbb{1} - \omega L^h D^{-1}$$

IS THE RESIDUAL AMPLIFICATION MATRIX. CLEARLY,

$$\boxed{\tilde{r}^{(k)} = G_R^{\,k}\,\tilde{r}^{(0)}}$$

- NOW, G_R HAS A COMPLETE SET OF ORTHOGONAL EIGENVECTORS

$$\phi_m, \quad m = 1 \ldots n-2$$

WITH CORRESPONDING EIGENVALUES μ_m. IN PARTICULAR, WE CAN EXPAND

$$\tilde{r}^{(0)} = \sum_{m=1}^{n-2} c_m \phi_m \rightarrow \tilde{r}^{(k)} = \sum_{m=1}^{n-2} c_m (\mu_m)^k \phi_m$$

("FOURIER" ANALYSIS

THUS, AFTER K SWEEPS, MTH (FOURIER) COMPONENT OF $\tilde{r}^{(0)}$ IS REDUCED BY A FACTOR OF $(\mu_m)^k$

AGAIN, FOR ILLUSTRATIVE PURPOSES, TAKE $\omega = \frac{1}{2}$, THEN CAN BE SHOWN THAT

$$\boxed{\begin{aligned} \phi_m &= \left(\sin(\pi m h),\ \sin(2\pi m h),\ \ldots,\ \sin((n-2)\pi m h)\right) \qquad (17a) \\ \mu_m &= \cos^2\left(\tfrac{1}{2}\pi m h\right) \qquad m = 1, 2, \ldots, n-2 \qquad (17b) \end{aligned}}$$

NOTE: EACH ϕ_m HAS ASSOCIATED "WAVELENGTH" λ_m

$\sin(\pi m x) \equiv \sin(2\pi x/\lambda_m) \rightarrow$

$\boxed{\lambda_m = \frac{2}{m}}$ AS M INCREASES, SO DOES FREQUENCY OF ϕ_m (λ_m DECREASES), μ_m DECREASES.

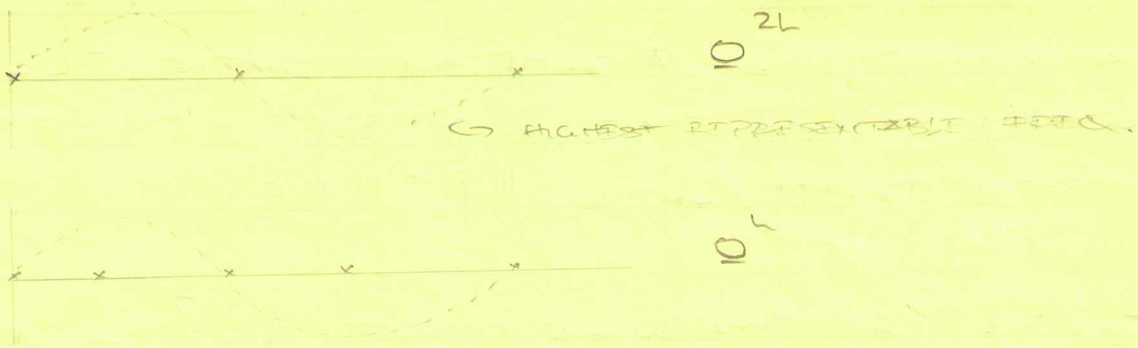
AS DISCUSSED PREVIOUSLY, ASYMPTOTIC CONVERGENCE RATE IS DETERMINED BY LARGEST μ_m, WHICH IS μ_1

$$\mu_1 = \cos^2\left(\tfrac{1}{2}\pi h\right) = 1 - \tfrac{1}{4}\pi^2 h^2 + \cdots = 1 - O(h^2)$$

WHICH IS THE SAME BASIC RESULT PREVIOUSLY QUOTED FOR GS → IMPLIES $O(n^2)$ SWEEPS TO REDUCE $\|\tilde{r}^{n}\|$ BY 10

THUS ASYMPTOTIC CONVERGENCE OF THIS (AND MANY OTHER) RELAXATION SWEEPS IS DOMINATED BY THE (SLOW) CONVERGENCE OF SMOOTH (LOW FREQUENCY, LARGE WAVELENGTH) COMPONENTS OF $\tilde{r}^{(n)}$ (ALSO $\tilde{u}^{(n)}$)

- WHAT HAPPENS TO HIGH FREQUENCY COMPONENTS? IN PARTICULAR, CONSIDER THOSE COMPONENTS OF $\tilde{r}^{(n)}$ WHICH CANNOT BE REPRESENTED ON A COARSER GRID Ω^{2h}



$\hookrightarrow \lambda = 4h$

THUS, WE ARE CONCERNED WITH ϕ_m SUCH THAT $\lambda_m \leq 4h \rightarrow mh \geq \frac{1}{2}$ IN THIS CASE

$$\mu_m = \cos^2\left(\tfrac{1}{2}\pi m h\right) \leq \cos^2\left(\tfrac{\pi}{4}\right) = \tfrac{1}{2}$$

$$\boxed{\mu_{HI} \leq \tfrac{1}{2}}$$

SO, ALL COMPONENTS WHICH CANNOT BE REPRESENTED ON Ω^{2h} GET SUPRESSED BY A FACTOR OF AT LEAST ½ PER SWEEP. FURTHERMORE, ~~RATE~~ AT WHICH HIGH-FREQUENCY COMPONENTS ARE LIQUIDATED IS INDEPENDENT OF MESH-SPACING, h.

SUMMARY: RELAXATION TENDS TO BE A DISMAL SOLVERS OF SYSTEMS $L^h u^h = f^h$ ARISING FROM FDAs OF ELLIPTIC PDEs; HOWEVER, IT TENDS TO BE A VERY GOOD SMOOTHER OF SUCH SYSTEMS

THE MULTI-GRID METHOD (ACHI BRANDT)

- CONSIDER AGAIN OUR MODEL PROBLEM

$$u_{xx} + u_{yy} = f(x,y) \text{ ON } [0,1]\times[0,1] \text{ WITH } u|_{\partial\Omega} = 0$$

DISCRETIZED USING THE STANDARD 5-PT FDA APPROX ON AN $n \times n$ GRID. ASSUME AN ITERATIVE SOL^N ALGORITHM → WILL START WITH SOME INITIAL ESTIMATE $\tilde{u}^{(0)}$, THEN ITERATE UNTIL $\|\tilde{r}^{(n)}\| \leq \epsilon$

QUESTIONS: (1) HOW DO WE CHOOSE n ?
(2) HOW DO WE CHOOSE ϵ ?
(3) HOW DO WE CHOOSE $\tilde{u}^{(0)}$?
(4) HOW FAST CAN WE "SOLVE" $L^h u^h = f^h$?

ANSWERS: (PARTIAL)

(1) CHOOSING n → CHOOSING h

- IDEALLY CHOOSE n SUCH THAT $\|u^h - u\| < \epsilon_u$
  $\epsilon_u \equiv$ USER-PRESCRIBED ERROR TOLERANCE
- WE ALREADY KNOW HOW TO ESTIMATE ERROR FOR ESSENTIALLY ANY F.D. SOL^N

$$u^h \sim u + h^2 e_2 + \ldots O(h^4) \quad \text{(RICHARDSON)}$$
$$u^{2h} \sim u + 4h^2 e_2 + \ldots$$

$$e \sim (3h^2)^{-1}(u^{2h} - u^h)$$

I.E. PERFORM CONVERGENCE TESTS → COMPARE FD

(2) CHOOSING ϵ (ϵ_R)

$$L^h u^h - f^h = 0$$

$$L^h \tilde{u}^h - f^h = \tilde{r}^h$$

$$L^h u - f^h = \tau^h \rightarrow \text{TRUNCATION ERROR}$$

NATURAL TO STOP WHEN $\|\tilde{r}^h\| \sim \|\tau^h\|$. WE WILL SEE HOW ESTIMATES OF τ^h ARISE NATURALLY IN MG.

(3) CHOOSING $\tilde{u}^{(0)}$

IDEA: USE SOL^N OF COARSE-GRID PROBLEM AS INITIAL ESTIMATE FOR FINE GRID PROBLEM

I.E. SOLVE $\boxed{L^{2h} u^{2h} = f^{2h}}$ (*), THEN SET

$$(u^h)^{(0)} := \mathbb{I}^h_{2h} u^{2h}$$

↳ COARSE-TO-FINE PROLONGATION OPERATOR

TYPICALLY $\mathbb{I}^h_{2h}$ WILL PERFORM d-DIMENSIONAL POLYNOMIAL INTERPOLATION TO A SUITABLE ORDER IN MESH-SPACING h { (LINEAR: $O(h^2)$, QUADRATIC: $O(h^3)$, AND CUBIC: $O(h^4)$ MOST COMMON CASES.) }
DEFER

COARSE SOL^N SHOULD BE INEXPENSIVE TO COMPUTE RELATIVE TO FINE (AT MOST $\left(\frac{1}{2}\right)^d$)

APPLY IDEA RECURSIVELY; INITIALIZE $\tilde{u}^{2h}$ WITH SOLUTION u^{4h} ETC.

PSEUDO-CODE: MULTILEVEL TECHNIQUE FOR ELLIPTIC PDES (BUT GENERALIZES TO OTHER CLASSES)

```
FOR l = 1, LEVEL
    IF l = 0 THEN
        u_l := 0
    ELSE
        u_l := Prolong(u_{l-1})
    END IF
    SOLVE_ITERATIVELY(u_l)
END FOR
```

(4) HOW FAST CAN WE SOLVE $L^h u^h = f^h$?

$\boxed{O(N) \text{ OPERATIONS}}$ MULTI-GRID PERFORMANCE
|||
COMPUTATIONALLY OPTIMAL

IMPORTANT PREAMBLE

(1) ASSUMPTION: (CONVENIENCE, SUFFICIENCY) USE "2:1" REFINEMENT EXCLUSIVELY → DISCRETIZATION SCALES

COARSEST

$$h_l, \; l = 1, 2, l_{max}$$

FINEST

$$h_{l+1} = \frac{1}{2} h_l$$

$$n_{l+1} \sim 2^d n_l = 4 n_l \quad \text{FOR M.P.}$$

(2) OBSERVATION: MG COST DOMINATED BY RELAXATION (SMOOTHING) SWEEPS, PER SWEEP WORK

$$\omega_\ell = 2^d \omega_{\ell-1} = 2^{2d} \omega_{\ell-2} \cdots$$

COARSE GRID WORK CHEAP

NOTATION

(0) CG = COARSE GRID CGC = CG CORRECTION

(1) $u^h, u^{2h}, \ldots, L^h, L^{2h}, \ldots \Pi^h_{2h}, I^h_{2h}, I^{2h}_h, \ldots$
(BASIC MG COMPONENTS ARE $h \to 2h$, $2h \to h$)

(2) u_ℓ, $u^\ell \equiv u^{h_\ell}$ (USEFUL IN PSEUDO-CODE)

(3) $L^h u^h = f^h \to L^h u^h = rhs^h$ WHERE rhs^h WILL SOMETIMES $= f^h$ (I.E. MG INVOLVES SOL^n OF CG PROBLEMS OTHER THAN $L^h u^h = f^h$)

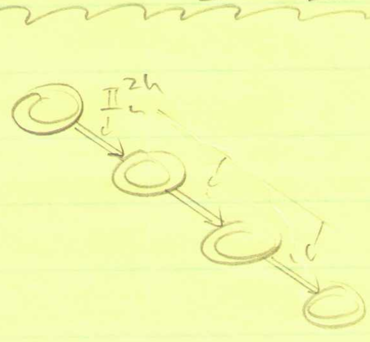
MULTI-LEVEL ALGORITHM REVISITED

$\ell = 1$

$\ell = 2$

$\ell = 3$

$\ell = 4$



$\odot$: $L^{h_\ell} u^{h_\ell} = f^{h_\ell}$ SOLVED

$$\| u^{h_\ell} - \tilde{u}^{h_\ell} \| \le \epsilon_u$$

ONE USE FOR "GRID HIERARCHY"

HOWEVER ...

MG ON LEVEL ℓ USES $\ell-1, \ell-2, \ldots,$ GRIDS FOR ANOTHER PURPOSE

> ACCELERATE CONVERGENCE USING CGs TO EFFECTIVELY AND CHEAPLY ANNIHILATE SMOOTH COMPONENTS OF $\tilde{r}^h, \tilde{e}^h$

MULTI-GRID FOR LINEAR PROBLEMS

LINEAR CORRECTION SCHEME (LCS) ALGORITHM (HW 4)

APPLY RBGS (SAY) TO $L^h u^h = f^h$ (THINK 2-D M.P.)
AFTER A FEW SWEEPS (PERHAPS 1!), RESIDUAL

$$\tilde{r}^h := L^h \tilde{u}^h - f^h \tag{1}$$

AND THE ERROR

$$\tilde{e}^h := u^h - \tilde{u}^h \tag{2}$$

WILL BE SMOOTH (AS WE HAVE SEEN)

COMPUTING SOL^N OF $L^h u^h = f^h$ ≡ COMPUTING CORRECTION v^h S.T.

$$u^h = \tilde{u}^h + v^h \tag{3}$$

CLEARLY, v^h SATISFIES

$$L^h v^h = -\tilde{r}^h \tag{4}$$

$\tilde{r}^h, v^h$ BOTH SMOOTH: CAN SENSIBLY POSE CG VERSION OF (4)

$$\boxed{\underline{L}^{2h}\, v^{2h} = -\, I_h^{2h}\, \tilde{r}^h \equiv rhs^{2h}} \qquad (5)$$

$\underline{L}^{2h}$ ≡ COARSE-GRID DIFFERENCE OPERATOR ("SAME" OP AS $\underline{L}^h$)

I_h^{2h} ≡ FINE-TO-COARSE TRANSFER OPERATOR; OFTEN CALLED RESTRICTION OPERATOR

NOW ASSUME SOME APPROX SOL^{N} $\tilde{v}^{2h}$ OF (5) IS COMPUTED; UPDATE $\tilde{u}^h$ VIA

$$\boxed{\tilde{u}^h := \tilde{u}^h + \underbrace{I_{2h}^{h}\, \tilde{v}^{2h}}_{\text{COARSE GRID CORRECTION}},} \qquad (6)$$

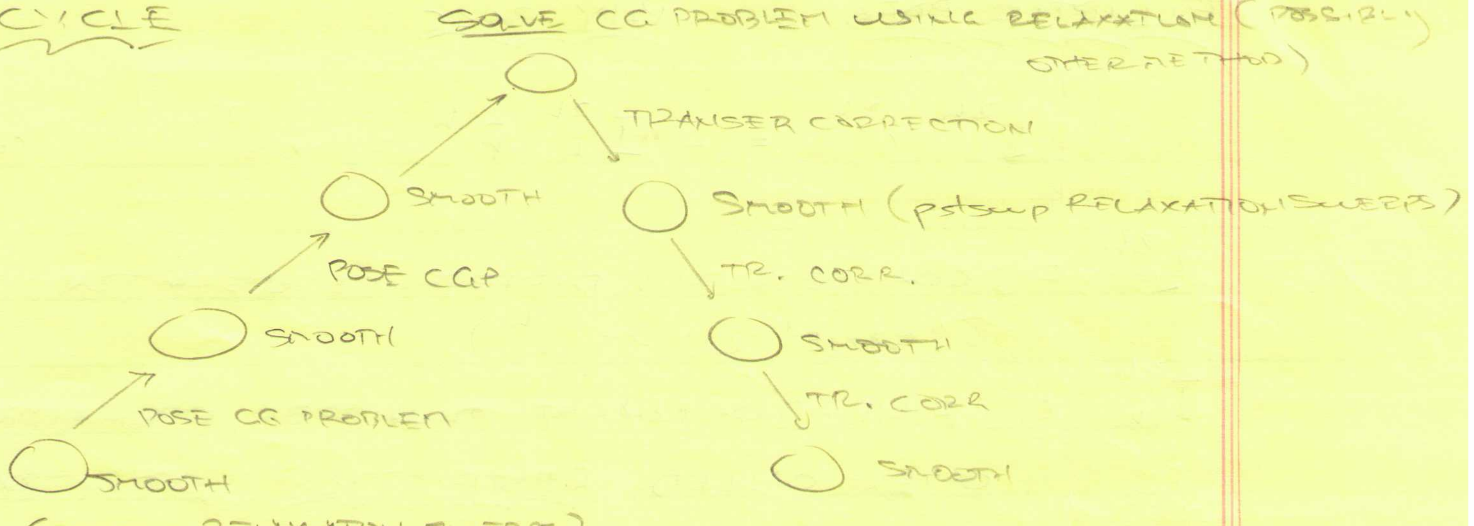
I_{2h}^{h} ≡ (ANOTHER) COARSE-TO-FINE TRANSFER OPERATOR, PROLONGATION OPERATOR (TYPICALLY POLYNOMIAL INTERPOLATION)

INTERPOLATION OF CGC → HIGH FREQ COMP IN $\tilde{r}^h$

APPLY POST-CGC SMOOTHING SWEEPS

- APPLY SMOOTH / CGC / SMOOTH PROCESS RECURSIVELY TO SOLVE COARSE GRID PROBLEMS ON 2h (5), 4h, ⋯ UNTIL ON COARSEST LEVEL $l=1$, (3×3 = 1 INTERIOR POINT), PROBLEM IS TRIVIAL TO SOLVE (NOT JUST SMOOTH) VIA RELAXATION

V-CYCLE

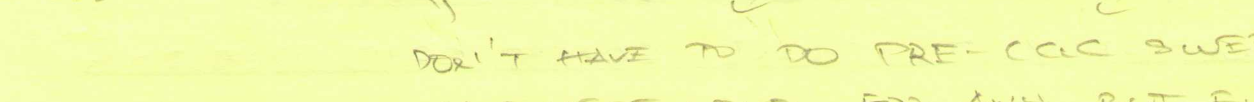


- ASSUMING INIT. ESTIMATE OF $\tilde{u}^h$ GOOD (MULTI-LEVEL STRATEGY), TYPICALLY FIND $\|\tilde{r}^h\| \sim \|\tau^h\|$ AFTER 1 V-CYCLE → PROBLEM EFFECTIVELY SOLVED

- ADD'L V-CYCLES WILL DRIVE $\|\tilde{r}^h\| \to 0$, TYPICALLY $\rho(G_{vcycle}) \le 0.1$ (ORDER OF MAG PER V-CYCLE) EXTRA CYCLES MAY BE NECESSARY IF WANT TO RICH. EXTRAP. FOR EXAMPLE

→ presmp, pstsmp SPECIFIED

LCS FIXED V-CYCLE PSEUDO CODE



DON'T HAVE TO DO PRE-CGC SWEEPS ON FINEST LEVEL FOR ANY BUT <u>FIRST</u> V-CYCLE.

```
PROCEDURE lcs_vcycle(l, cycle, preswp, pstswp)
  DO m = l, 2, -1
     IF cycle = 1 OR m ≠ l THEN
        DO preswp TIMES
           u^m := relax_rb(u^m, rhs^m, h_m)
        END DO
     END IF
     u^(m-1) := 0
     rhs^(m-1) := -I_m^(m-1) (L^m u^m - rhs^m)
  END DO
  u^1 := relax_rb(u^1, rhs^1, h_1) UNTIL ||r̃^1|| ≤ ε_1
  DO m = 2, l, +1
     u^m := u^m + I_(m-1)^m u^(m-1)
     DO pstswp TIMES
        u^m := relax_rb(u^m, rhs^m, h_m)
     END DO
  END DO
END PROCEDURE
```

TRANSFER OPERATORS

I_m^{m-1} → 2:1 HALF-WEIGHTED RESTRICTION

I_{m-1}^m → 2:1 (MULTI-) LINEAR INTERPOLATION

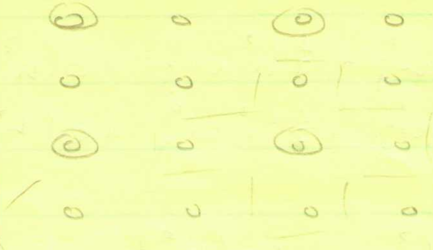
DRIVER ROUTINE (PERFORMS MULTI-LEVEL ALG)

```
PROCEDURE mglcs(lev, ncycle, preswp, pstswp)
  DO l = 1, lev
     IF l = 1 THEN
        u^l := 0
     ELSE                    -> 2-D - BI-LINEAR INTERP
        u^l := I^l_{l-1} u^{l-1}
     END IF
     rhs^l := f^l
     DO cycle = 1, ncycle
        lcs_vcycle(l, cycle, preswp, pstswp)
     END DO
  END DO
END PROCEDURE
```

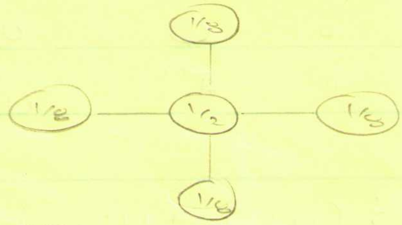
GRID-TO-GRID TRANSFER OPERATORS : NON-TRIVIAL INTERACTION WITH RELAXATION (SMOOTHING) SCHEME, ORDER - PART. RB VS LEX.

HALF-WEIGHTED RESTRICTION I_h^{2h}, $I_\ell^{\ell-1}$

- USE FOR MODEL PROBLEM (IN 1, 2, 3-D) WHEN RC/BS IS SMOOTHER

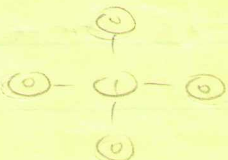


→ USE 5-PT TEMPLATE



NEED TO DEFINE THESE VALUES

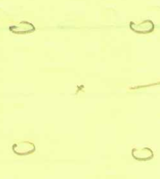
IF USING LEX GS, INJECTION MAY WORK BETTER



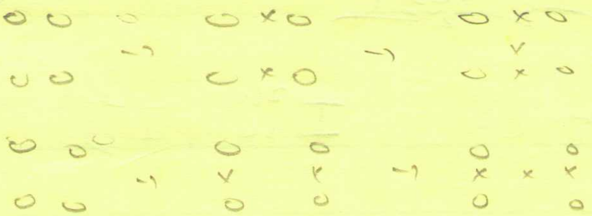
PROLONGATION: BI-LINEAR INTERPOLATION I_{2h}^{h}, $I_{\ell-1}^{\ell}$

ORDER	NAME	TRUNC. ERROR
1	"NEAREST VALUE"	$O(h)$
2	LINEAR	$O(h^2)$
3	QUADRATIC	$O(h^3)$
4	CUBIC	$O(h^4)$

BI-LINEAR: INDEPENDENT 1-D LINEAR INTERP IN X AND Y DIRECTIONS



→ NO UNIQUE $O(h^2)$ VALUE



SHOULDN'T; DOESN'T MATTER

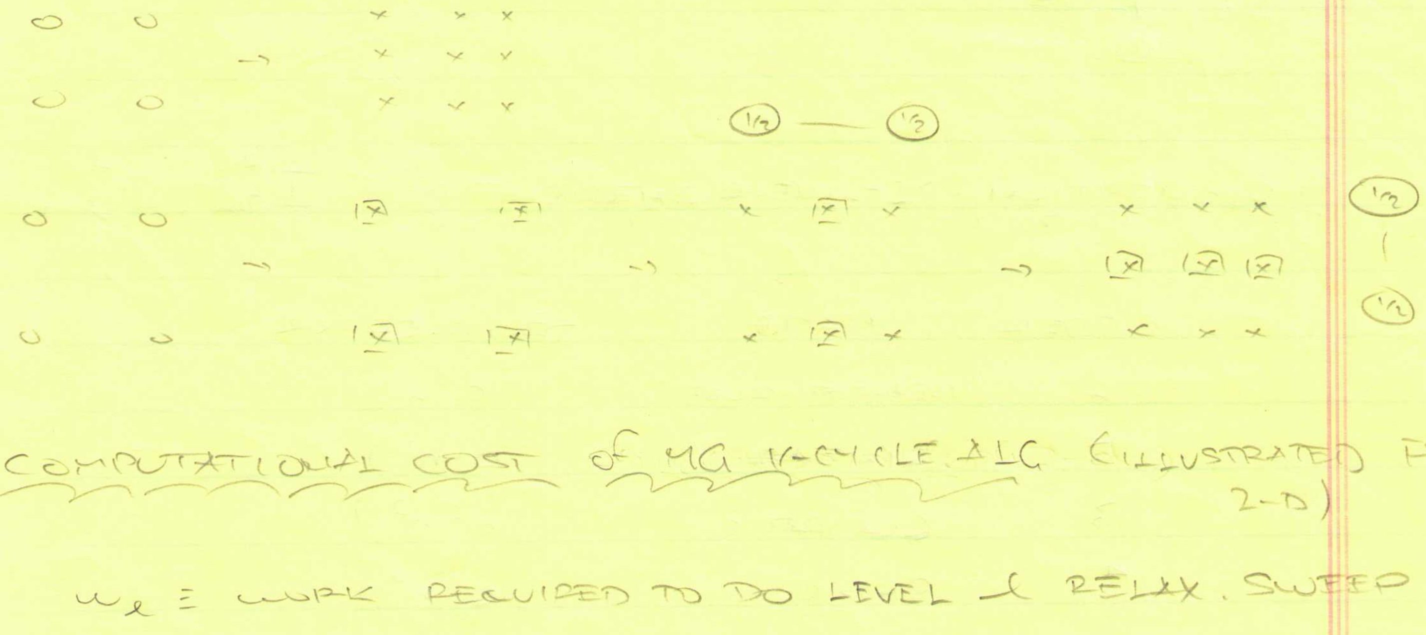
SPECIAL CASE: 2:1 LINEAR INTERPOLATION

$$\begin{array}{ccc} \bigcirc & \circ & \bigcirc \\ j-1 & j & j+1 \end{array} \qquad \begin{aligned} u_{j+1} &= u_j + h u_j' + \tfrac{1}{2} h^2 u_j'' + O(h^3) \\ u_{j-1} &= u_j - h u_j' + \tfrac{1}{2} h^2 u_j'' + O(h^2) \end{aligned}$$

$$\to \tfrac{1}{2}\left(u_{j+1} + u_{j-1}\right) = u_j + \tfrac{1}{2} h^2 u_j'' + O(h^4)$$

$$= u_j + O(h^2)$$

IMPLEMENTATION TRICK (GENERALIZES TO $d=3$, HIGHER ORDER)



COMPUTATIONAL COST of MG V-CYCLE ALG (ILLUSTRATED FOR 2-D)

$w_\ell \equiv$ WORK REQUIRED TO DO LEVEL ℓ RELAX. SWEEP

$W_\ell \equiv$ " " " SOLVE LEVEL ℓ PROBLEM

$p \equiv$ presweep $\quad q \equiv$ postswp $\quad r \equiv$ # of CGCs PER SOLVE

$$W_\ell \simeq (p+q)\, w_\ell + r\, W_{\ell-1}$$

$$= (p+\sigma q)\,\omega_\ell + \sigma\left((p+\sigma q)\,\omega_{\ell-1} + \sigma \bar{W}_{\ell-2} \right)$$

BUT $\omega_{\ell-1} = \frac{1}{4}\omega_\ell = 2^{-d}\omega_\ell$

$$= (p+\sigma q)\,\omega_\ell \left(1 + \frac{1}{4}\sigma + \frac{1}{16}\sigma^2 + \cdots + \left(\frac{\sigma}{4}\right)^{\ell-1} \right) + \sigma^\ell \bar{W}_0$$

$\bar{W}_0 \equiv$ WORK REQUIRED TO SOLVE $L^0 u^0 = rhs^0$

IF $\sigma < 4$, $1 + \frac{1}{4}\sigma + \cdots < \left(1 - \frac{\sigma}{4}\right)^{-1}$

$$\Rightarrow \bar{W}_\ell \le \omega_\ell \left[\frac{p+\sigma q}{1 - \sigma/4} \right] + \sigma^\ell \bar{W}_0$$

BUT $\omega_\ell \sim c\, n_\ell$ $\quad \sigma^\ell \le 4^\ell \simeq \frac{n_\ell}{n_0}$

$$\boxed{\Rightarrow \bar{W}_\ell \le n_\ell \left[\frac{c(p+\sigma q)}{1 - \sigma/4} + \frac{\bar{W}_0}{n_0} \right] = O(n_\ell)}$$

AS ADVERTISED!

MULTIGRID FOR NON-LINEAR PROBLEMS

FULL APPROXIMATION STORAGE (FAS) ALGORITHM

(1) LCS RECAP

$$L^h u^h - f^h = 0 \qquad L^h \tilde{u}^h - f^h = \tilde{r}^h$$

ASSUMING SMOOTH(ED) $\tilde{r}^h$, SOUGHT CORRECTION v^h

$$u^h = \tilde{u}^h + v^h$$

COMPUTED ON COARSER MESH BY SOLVING / UPDATING

$$L^{2h} u^{2h} = - I_h^{2h} r^h \qquad \tilde{u}^h := \tilde{u}^h + I_{2h}^h v^{2h}$$

BUT WE HAD TO USE LINEARITY OF L^h

$$L^h(\tilde{u}^h + v^h) = L^h \tilde{u}^h + L^h v^h$$

WON'T WORK IF L^h NON-LINEAR. HOWEVER, SUPPOSE WE CAN STILL SMOOTH $\tilde{r}^h$. THEN CAN AGAIN VIEW SOLN IN TERMS OF SMOOTH CORRECTION v^h SUCH THAT

$$L^h(\tilde{u}^h + v^h) = f^h$$

IN PARTICULAR, CONSIDER

$$L^h(\tilde{u}^h + v^h) - L^h \tilde{u}^h = f^h - L^h \tilde{u}^h = -\tilde{r}^h$$

PROVIDED $\tilde{r}^h$ IS SMOOTH, CAN SENSIBLY POSE CG VERSION OF (1), NAMELY

$$L^{2h} u^{2h} - L^{2h} I_h^{2h} \tilde{u}^h = - I_h^{2h} r^h$$

$$\boxed{L^{2h} u^{2h} = L^{2h} I_h^{2h} \tilde{u}^h - I_h^{2h} r^h} \quad (1)$$

CG REP^N OF $\tilde{u}^h + v^h$ → "FULL APPROXIMATION", HENCE FAS.

NOW, AS WITH LCS, SOLVE CG PROBLEM, THEN UPDATE FINE GRID EQN VIA

SOLN OF CG PROBLEM

$$\boxed{\tilde{u}^h := \tilde{u}^h + I_{2h}^h \left(u^{2h} - I_h^{2h} \tilde{u}^h \right)} \quad (2)$$

WHICH IS NOT EQUIVALENT TO THE MORE OBVIOUS

$$\tilde{u}^h := I_{2h}^h u^{2h}$$

SINCE $I_{2h}^h I_h^{2h} \neq 1$ IN GENERAL (RANK DEFICIENCY).
FORMER PREFERABLE SINCE "RETAINS (USEFUL) HIGH FREQ. INFO CONTAINED IN $\tilde{u}^h$"

EQN^S (1) AND (2) ARE BASIC EQN^S FOR FAS-CGC

CONSIDER ALTERNATE DERIVATION of (1) WHICH REVEALS ANOTHER WAY OF VIEWING MG.

<u>RECALL</u>: $Lu - f = 0$ CONTINUUM

$L^h u^h - f^h = 0$ DISCRETE (DIFFERENCE

LOCAL TRUNCATION ERROR : $\tau^h \equiv L^h u - f$ (3)

$$\rightarrow L^h u = f + \tau^h$$

SUPPOSE WE HAD AN APPROXIMATION OF τ^h, $\tilde{\tau}^h$

THEN BY SOLVING

$$L^h u^h_{IMP} = f + \tilde{\tau}^h$$

WE WOULD HAVE AN IMPROVED APPROX u^h_{imp} RELATIVE TO u^h (I.E. $\|u^h_{imp} - u\| < \|u^h - u\|$

VIEW TRUNCATION ERROR AS THAT QUANTITY WHICH WHEN ADDED TO THE rhs OF A DIFFERENCE EQ^N (THE FORCING TERMS), MAKES DIFFERENCE SOL^N COINCIDE WITH CONTINUUM SOL^N

(3) CAN ALSO BE REWRITTEN :

$$\tau^h = L^h u - L^h u^h$$

RE-INTRODUCE 2h MESH AND CONSIDER

$$\boxed{\tau^{2h}_h \equiv L^{2h} I^{2h}_h u^h - I^{2h}_h L^h u^h} \quad (3)$$

τ^{2h}_h = RELATIVE (LOCAL) TRUNCATION ERROR
(DEFINED ON 2h, RELATIVE TO h-GRID)

$$\rightarrow L^{2h} \underline{I_h^{2h}} u^h = I_h^{2h} L^h u^h + \tau_h^{2h}$$

$$\boxed{L^{2h} u^{2h} = f^{2h} + \tau_h^{2h}} \qquad (4)$$

SO: (IN DIRECT ANALOGY TO τ^h) CAN THINK OF τ_h^{2h} AS CORRECTION TO "FORCING TERM" (RHS) OF CG EQN^S WHICH MAKES SOL^N OF CG EQNS = SOL^N OF FINE GRID EQN^S

IN GENERAL, CAN'T COMPUTE τ_h^{2h} EXACTLY, BUT CAN APPROXIMATE IT:

$$\tau_h^{2h} \simeq \boxed{\tilde{\tau}_h^{2h} = L^{2h} I_h^{2h} \tilde{u}^h - I_h^{2h} L^h \tilde{u}^h} \qquad (5)$$

THEN (4) BECOMES

$$L^{2h} u^{2h} = f^{2h} + \tilde{\tau}_h^{2h} = f^{2h} + L^{2h} I_h^{2h} \tilde{u}^h - I_h^{2h} L^h \tilde{u}^h$$

$$= L^{2h} I_h^{2h} \tilde{u}^h - I_h^{2h} \left(L^h \tilde{u}^h - f^h \right)$$

$$= L^{2h} I_h^{2h} \tilde{u}^h - I_h^{2h} \tilde{r}^h \quad \rightarrow \text{EQN } (1)$$

THUS WE HAVE DUAL VIEWS OF MG PROCESS

I) CG USED TO ACCELERATE CONV of LOW-FREQ COMP. OF FG RESIDS.

II) FGs USED TO COMPUTE CORRECTION TERMS TO CG EQN^S, YIELDING FG ACCURACY ON CGs.

(4) AND (5) (WITH $\tau_h^{2h} \rightarrow \tilde{\tau}_h^{2h}$) ARE EQUIVALENT TO (1) AND (2); FORM WE WILL USE IN IMPLEMENTATION

ESTIMATING TRUNCATION ERROR

(1) $\|\tau^h\|$ PROVIDES NATURAL STOPPING CRITERIA
$\|\tilde{r}^h\| \sim \|\tau^h\|$

(2) TAU-EXTRAPOLATION: INCREASE ACCURACY OF u^h BY SOLVING $L^h u^h = f^h + \tilde{\tau}^h$

CAN SHOW THAT FOR $O(h^2)$ L^h AND 2:1 REFINEMENT

$$\boxed{\tau^h \sim \frac{1}{3}\tau_h^{2h} \sim \frac{1}{3}\tilde{\tau}_h^{2h}}$$

NOTE: CAN USE $\|\tilde{\tau}^{2h}\|$ AS NATURAL STOPPING CRITERION FOR "ADAPTIVE" MG.

NON-LINEAR RELAXATION / SMOOTHING

- FOR "TYPICAL" NON-LINEAR PROBLEMS (CARTESIAN COORDS; SMOOTHLY VARYING, WELL-SCALED COEFFICIENTS), 1-STEP NEWTON RBGS IS A GOOD SMOOTHER

EXAMPLE: $\Delta u + \sigma u^2 = f(x,y)$

$$F[u_{ij}] \equiv h^{-2}(\tilde{u}_{i+1,j} + \tilde{u}_{i-1,j} + \tilde{u}_{i,j+1} + \tilde{u}_{i,j-1} - 4\tilde{u}_{i,j}) + \sigma\tilde{u}_{ij}^2 - f_{ij} = 0$$

$$J_D[u_{ij}] \equiv \frac{\partial F[u_{ij}]}{\partial u_{ij}} = -4h^{-2} + 2\sigma u_{ij}$$

1-STEP NEWTON GS

$$\boxed{\tilde{u}_{ij} := \tilde{u}_{ij} - \frac{F[\tilde{u}_{ij}]}{J_D[\tilde{u}_{ij}]}}$$

TAKE ONLY ONE NEWTON STEP/VISIT.

PROCEDURE fas_vcycle(l, cycle, preswp, pstswp)

Do $m = l, 2, -1$

IF cycle $= 1$ OR $m \neq l$ THEN

DO PRESWP TIMES

$u^m := \text{relax\_rb}(u^m, rhs^m, h_m)$

END DO

END IF

$\tau_m^{m-1} = L^{m-1} I_m^{m-1} u^m - I_m^{m-1} L^m u^m$

$rhs^{m-1} := \tau_m^{m-1} + I_m^{m-1} rhs^m \; ; \; u^{m-1} := I_m^{m-1} u^m$

END DO

$u^1 := \text{relax\_rb}(u^1, rhs^1, h_1)$ UNTIL $\|\tilde{r}^1\| < \epsilon$

Do $m = 2, l, +1$

$u^m := u^m + I_{m-1}^m (u^{m-1} - I_m^{m-1} u^m)$

DO PSTSWP TIMES

$u^m := \text{relax\_rb}(u^m, rhs^m, h_m)$

END DO

END DO

END PROCEDURE

MAIN ROUTINE (mgfas) ESSENTIALLY IDENTICAL TO LCS MAIN ROUTINE

ENHANCEMENTS