Constraint Preserving Boundary Conditions in Numerical Relativity

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OUTLINE

- FOSH Systems with Constraints
- Boundary Conditions for FOSH Systems
- Constraint Preserving Boundary Conditions for a Model Problem
- Boundary Conditions for Einstein's Equations
- Conclusions and Future Directions

FOSH Systems with Constraints

Find $u(t,x) \in \mathbf{R}^m, x \in \Omega \subset \mathbf{R}^n, t \in [0, T]$ such that:

$$\dot{u} = Au + f$$
$$u(0,x) = u_0(x) \qquad \text{and} \qquad C := Bu = 0 \quad \text{(constraints)}$$
$$\text{boundary conditions}$$

$A = \sum_{i=1}^n A^i \partial_i$ is a first order differential operator, where the $m \times m$ matrices A^i are constant and symmetric.

B is some $k \times m$ first order differential operator. Of course, we assume that $Bu_0 = 0$, and $Bf = 0$. Also, that the null space of B is invariant under A.

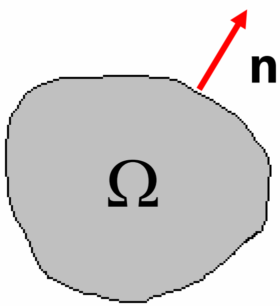
The question is what boundary conditions make the problem well posed and preserve the constraints.

Maximal Nonnegative Boundary Conditions

Boundary Conditions: $u(t,x) \in N(t,x)$, $\forall (t,x) \in [0,T] \times \partial\Omega$.

These boundary conditions are called maximal nonnegative if the boundary matrix $A_n(x) = -\sum_{i=1}^{n} n_i(x) A^i$ is nonnegative over $N(t,x)$

$$u^T A_n(x) u \geq 0, \ \forall u \in N(t,x)$$



and $N(t,x)$ is maximal with this property. Observe that the dimension of $N(t,x)$ must be equal to the number of positive and null eigenvalues counted with their multiplicities.

It is well known that FOSH systems with maximal nonnegative boundary conditions are well posed.

[Friedrichs, Lax, Plillips, Kreiss, Rauch, Majda, Osher, Higdon, Secchi, etc.]

Maximal Nonnegative Boundary Conditions

Suppose $A_n(x)$ has:
l_0 null eigenvalues $(\lambda_1, e_1^\alpha), \ldots, (\lambda_{l_0}, e_{l_0}^\alpha)$,
l_- negative eigenvalues $(\lambda_{l_0+1}, e_{l_0+1}^\alpha), \ldots, (\lambda_{l_0+l_-}, e_{l_0+l_-}^\alpha)$,
l_+ positive eigenvalues $(\lambda_{l_0+l_-+1}, e_{l_0+l_-+1}^\alpha), \ldots, (\lambda_m, e_m^\alpha)$.
Correspondingly:
l_- incoming characteristic fields $u_j^- = e_j^\alpha u_\alpha$, $j = l_0+1, \ldots, l_0+l_-$,
l_+ outgoing characteristic fields $u_j^+ = e_j^\alpha u_\alpha$, $j = l_0+l_-+1, \ldots, m$.

Theorem: $N(t,x)$ is maximal nonnegative if and only if there exists a $l_- \times l_+$ matrix $M(t,x)$ with

$$|| \begin{pmatrix} \sqrt{|\lambda_{l_0+1}|} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sqrt{|\lambda_{l_0+l_-}|} \end{pmatrix} M(t,x) \begin{pmatrix} 1/\sqrt{\lambda_{l_0+l_-+1}} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \vdots & 1/\sqrt{\lambda_m} \end{pmatrix} || \leq 1$$

such that $N(t,x) = \{u \in \mathbf{R}^m : u^- = M(t,x)u^+\}$. [Kreiss, Lorenz, 1989]

Finding Constraint-preserving BC

Suppose that the constraints satisfy a FOSH system $\dot{C} = \Sigma_{i=1}^{n} D^i \partial_i C$.

- Write the general form of maximal nonnegative boundary conditions for the system of constraints.
- Trade the transverse derivatives for temporal and tangential ones using the main system.
- Equate the expressions with zero.
- Match these conditions with maximal nonnegative boundary conditions for the main system.

[Stewart, 1998; Friedrich, Nagy, 1999; Calabrese, Lehner, Tiglio, 2002; Bardeen, Buchman, 2002; Szilagyi, Winicour, 2003; Calabrese, Pullin, Reula, Sarbach, Tiglio, 2003; Calabrese, Sarbach, 2003; Frittelli, Gomez, 2003; Holst, Lindblom, Owen, Pfeiffer, Scheel, Kidder, 2004; Lindblom, Scheel, Kidder, Pfeiffer, Shoemaker, Teukolsky, 2004; Alekseenko, 2004; Kidder, Lindblom, Scheel, Buchman, Pfeiffer, 2004; Sarbach, Tiglio, 2004; among others.]

Extended System

Theorem: If $u_0 \in H^s(\mathbf{R}^n;\mathbf{R}^m)$ and $f(t,\,\cdot\,) \in H^s(\mathbf{R}^n;\mathbf{R}^m)$, $\forall t \geq 0$, for $s > m + n/2$, then $(u,z) \in C^1(\mathbf{R}^n \times [0,\infty);\mathbf{R}^{m+k})$ is a solution of the problem

$$\dot{u} = Au - B^*z + f, \quad \dot{z} = Bu, \quad u(0,x) = u_0(x), \quad z(0,x) = 0,$$

if and only if $z = 0$ and $u \in C^1(\mathbf{R}^n \times [0,\infty);\mathbf{R}^m)$ satisfies

$$\dot{u} = Au + f, \quad Bu = 0, \quad u(0,x) = u_0(x).$$

We are interested in a similar result on bounded domains.

On Bounded Domains

Original IBVP: $\dot{u} = Au + f,\ Bu = 0,\ u(0,x) = u_0(x),$
$u(t,x) \in N(x)\ (t \in [0,\ T],\ x \in \partial\Omega).$

Extended IBVP: $\dot{u} = Au - B^* z + f,\ \dot{z} = Bu,\ u(0,x) = u_0(x),\ z(0,x) = 0,$
$(u,z)(t,x) \in \overline{N}(x)\ (t \in [0,\ T],\ x \in \partial\Omega).$

Operators: $A := \Sigma_{i=1}^n A^i \partial_i,\ B := \Sigma_{i=1}^n B^i \partial_i,\ B^* := -\Sigma_{i=1}^n (B^i)^T \partial_i.$

Boundary matrices: $A_n(x) := -\Sigma_{i=1}^n n_i(x) A^i,\ B_n(x) := -\Sigma_{i=1}^n n_i(x) B^i,$

$$\overline{A}_n(x) = \begin{pmatrix} A_n(x) & B_n^T(x) \\ B_n(x) & 0 \end{pmatrix}.$$

Theorem: The boundary subspace $\overline{N} = N \times [B_n(N)]^\perp$ is nonnegative for $\overline{A}_n$ if and only if N is nonnegative for A_n. If both $\overline{N}$ and N are maximal nonnegative for $\overline{A}_n$ and A_n, respectively, then u is the solution of the original IBVP if and only if $(u,\, 0)$ is the solution of the extended IBVP.

Model Problem

We are interested in finding a solution (w_i, v_i, u_{ij}) for the following FOSH system in $\mathbf{R}^N$:

$$\dot{w}_i = v_i + g_i, \quad \dot{v}_i = \partial^j u_{ij} + f_i, \quad \dot{u}_{ij} = \partial_j v_i + h_{ij},$$

with initial data

$$w_i(0) = w_i^0, \quad v_i(0) = v_i^0, \quad u_{ij}(0) = u_{ij}^0,$$

and subject to the constraints

$$C := \partial^i v_i = 0, \quad C_j := \partial^i u_{ij} = 0.$$

The initial data and forcing terms satisfy the compatibility conditions:

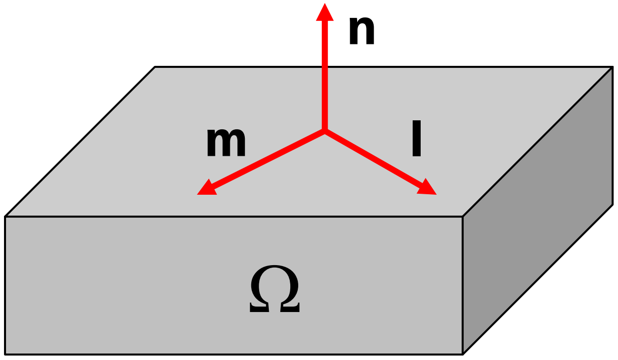
$$\partial^i v_i^0 = 0, \quad \partial^i u_{ij}^0 = 0, \quad \partial^i f_i = 0, \quad \partial^i h_{ij} = 0.$$

Since $\boxed{\dot{C} = \partial^j C_j,\ \dot{C}_j = \partial_j C}$, $C(0) = 0$, and $C_j(0) = 0$ the constraints are satisfied for all time for the pure Cauchy problem.

Boundary Conditions on Polyhedral Domains

A set of maximal nonnegative constraint-preserving boundary conditions is

$$n^i n^j u_{ij} = 0,\ \ \tau^i_j v_i = 0,$$



where n^i are the components of the unit normal and $\boxed{\tau^i_j := \delta^i_j - n^i n_j}$ is the projection operator orthogonal to the unit normal.

Theorem: Given initial conditions $w_i(0)$, $v_i(0)$, $u_{ij}(0)$ and forcing terms g_i, f_i, h_{ij} satisfying the compatibility conditions, define w_i, v_i, u_{ij} for positive times by the evolution equations and the above boundary conditions. Then, the constraints $C := \partial^i v_i = 0$ and $C_j := \partial^i u_{ij} = 0$ are satisfied for all time.

Strategy

- Write the general form of maximal nonnegative boundary conditions for the system of constraints:

$$\dot{C} = \partial^j C_j,\ \dot{C}_j = \partial_j C$$

- Trade the transverse derivatives for temporal and tangential ones using the main system.
- Equate the expressions with zero.
- Match these conditions with maximal nonnegative boundary conditions for the main system.

Idea of Proof

First, suppose that the forcing terms g_i, f_i, and h_{ij} vanish

$$\dot{w}_i = v_i, \quad \dot{v}_i = \partial^j u_{ij}, \quad \dot{u}_{ij} = \partial_j v_i.$$

- C and C_j satisfy the FOSH system $\dot{C} = \partial^j C_j$, $\dot{C}_j = \partial_j C$.
- $C(0) = 0, \; C_j(0) = 0.$
- $C = 0$ on the boundary due to the boundary conditions:

$$\begin{aligned} C &= \partial^i v_i = \delta^{ij}\partial_j v_i = (n^i n^j + \tau^{ij})\partial_j v_i = n^i n^j \partial_j v_i + \tau^{ij}\partial_j v_i \\ &= n^i n^j \dot{u}_{ij} + \tau^i_k \tau^{kj}\partial_j v_i = n^i n^j \dot{u}_{ij} + \tau^{kj}\partial_j(\tau^i_k v_i) - \tau^{kj}(\partial_j \tau^i_k) v_i \\ &= [n^i n^j u_{ij} + (\dot{N} - 1) H n^i w_i] + \tau^{kj}\partial_j(\tau^i_k v_i) = 0. \end{aligned}$$

- Thus, C and C_j vanish for all time.

Idea of Proof

To extend to the case of general g_i, f_i, and h_{ij} we use Duhamel's principle. Define

$$S(t)(w_i(0), v_i(0), u_{ij}(0)) = (w_i(t), v_i(t), u_{ij}(t)),$$

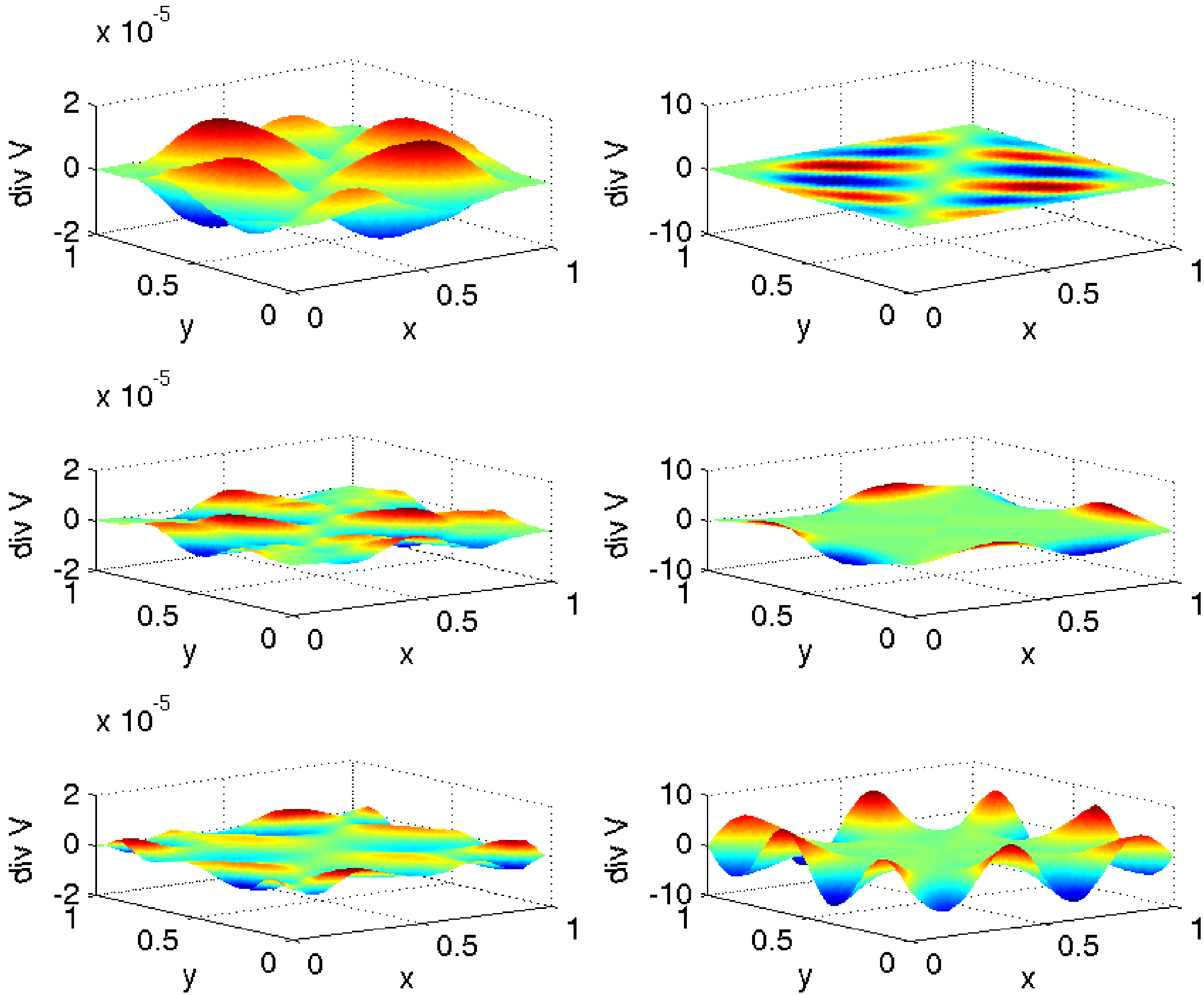
where (w_i, v_i, u_{ij}) is the solution of the homogeneous evolution equations

$$\dot{w}_i = v_i, \quad \dot{v}_i = \partial^j u_{ij}, \quad \dot{u}_{ij} = \partial_j v_i,$$

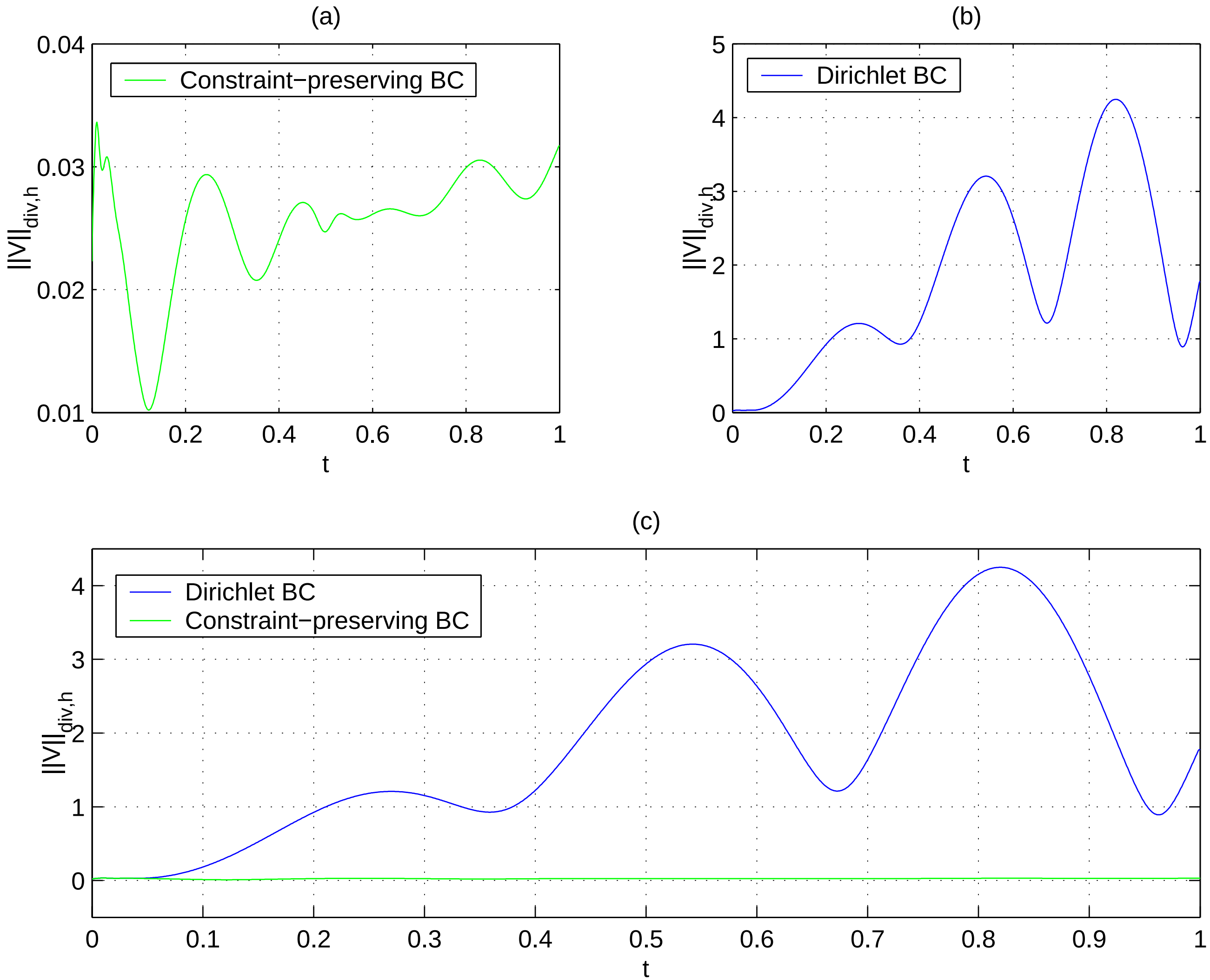
satisfying the boundary conditions and assuming the given initial data.
The solution of the inhomogeneous initial boundary value problem is

$$(w_i(t), v_i(t), u_{ij}(t)) = S(t)(w_i(0), v_i(0), u_{ij}(0)) + \int_0^t S(t-s)(g_i(s), f_i(s), h_{ij}(s))ds.$$

The integrand satisfies the constraints by the result for the homogeneous case, as does the first term on the right-hand side, and thus the constraints are satisfied.



[Arnold, Wang, 2003]



[Arnold, Wang, 2003]

BC for the Extended System

$$\dot{w}_i = v_i + g_i, \ \dot{v}_i = \partial^j u_{ij} + \partial_i p + f_i, \ \dot{u}_{ij} = \partial_j v_i + \partial_j q_i + h_{ij}, \ \dot{p} = \partial^i v_i, \ \dot{q}_i = \partial^i u_{ij}$$

A set of maximal non-negative constraint-preserving boundary conditions for this system is

$$n^i n^j u_{ij} = 0, \quad \tau_j^i v_i = 0, \quad p = 0, \quad \tau_j^i q_i = 0,$$

where n^i are the components of the unit normal and $\tau_j^i := \delta_j^i - n^i n_j$ is the projection operator orthogonal to the unit normal.

Theorem: Given initial conditions $w_i(0)$, $v_i(0)$, $u_{ij}(0)$ and forcing terms g_i, f_i, h_{ij} satisfying the compatibility conditions,

- set $p(0) = 0$, $q_i(0) = 0$
- evolve w_i, v_i, u_{ij}, p, q_i with boundary conditions

Then, p and q_i vanish for all time and w_i, v_i, u_{ij} satisfy the original problem.

The ADM Decomposition

The initial metric g_{ij} is partitioned into the lapse (a scalar) N, the shift (3D-vector) β_i, and the 3×3 spatial metric γ_{ij}

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

or

$$\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \left(\begin{array}{c|ccc} |\beta|^2 - N^2 & \beta_1 & \beta_2 & \beta_3 \\ \hline \beta_1 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \beta_2 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \beta_3 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{array}\right)$$

This partitioning divides Einstein's equations into two parts (Arnowitt, Deser, Misner '62):

The ADM Decomposition

Twelve evolution equations:

$G_{ij} = 0$

$$\dot{\gamma}_{ij} = -2NK_{ij} + \nabla_i \beta_j + \nabla_j \beta_i$$

$$\dot{K}_{ij} = N(R_{ij} - 2K_{il}K^l_j + KK_{ij}) + \beta^l \nabla_l K_{ij} - \nabla_i \nabla_j N + K_{il} \nabla_j \beta^l + K_{jl} \nabla_i \beta^l$$

Four constraint equations:

The Hamiltonian constraint: $R + K^2 - K^{ij}K_{ij} = 0$

$G_{00} = 0$

The momentum constraints: $\nabla_j (K^{ij} - \gamma^{ij} K) = 0$

$G_{0j} = 0$

Here, R_{ij} is Ricci's tensor, R is the scalar curvature, ∇_j is the covariant derivative, K_{ij} is the extrinsic curvature, and $K = \gamma^{ij} K_{ij}$ is its trace.

Linearized ADM

Look for solution as a perturbation of Minkowski's spacetime, unit lapse, and zero shift:

$$\gamma_{ij} = \delta_{ij} + \tilde{\gamma}_{ij}, \quad K_{ij} = 0 + \tilde{K}_{ij}, \quad N = 1 + \tilde{N}, \quad \beta_i = 0 + \tilde{\beta}_i.$$

In vector/matrix notation, to first order, $\underline{\underline{\gamma}}$, $\underline{\underline{K}}$, N, $\underline{\beta}$ satisfy:

$$\begin{aligned}
\dot{\underline{\underline{\gamma}}} &= -2\underline{\underline{K}} + 2\underline{\underline{\epsilon}}\underline{\beta} \\
\dot{\underline{\underline{K}}} &= \underline{\underline{R}}\underline{\underline{\gamma}} - \underline{\underline{\nabla}}\underline{\nabla} N \\
\operatorname{div}\underline{M}\underline{\underline{\gamma}} &= 0 \\
\underline{M}\underline{\underline{K}} &= \underline{0}
\end{aligned}$$

The initial data $\underline{\underline{\gamma}}(0) = \underline{\underline{\gamma}}_0$ and $\underline{\underline{K}}(0) = \underline{\underline{K}}_0$ is given.

$$\underline{\underline{\epsilon}}\underline{\beta} := \frac{1}{2}[\underline{\underline{\nabla}}\underline{\beta} + (\underline{\underline{\nabla}}\underline{\beta})^T], \quad \underline{\underline{R}}\underline{\underline{\gamma}} := \underline{\underline{\epsilon}}\underline{\operatorname{div}}\underline{\underline{\gamma}} - \frac{1}{2}\underline{\underline{\Delta}}\underline{\underline{\gamma}} - \frac{1}{2}\underline{\underline{\nabla}}\underline{\nabla}\operatorname{tr}\underline{\underline{\gamma}}, \quad \underline{M}\underline{\underline{\gamma}} := \underline{\operatorname{div}}\underline{\underline{\gamma}} - \underline{\nabla}\operatorname{tr}\underline{\underline{\gamma}}$$

FOSH Formulation (Arnold)

$$
\begin{aligned}
\dot{\underline{\underline{\gamma}}} &= -2\underline{\underline{K}} + 2\underline{\underline{\epsilon}}\underline{\beta} \\
\dot{\underline{\underline{K}}} &= \underline{\underline{R}}\underline{\underline{\gamma}} - \underline{\underline{\nabla}}\underline{\nabla} N \\
\mathrm{div} \underline{M} \underline{\underline{\gamma}} &= 0 \\
\underline{M} \underline{\underline{K}} &= \underline{0}
\end{aligned}
$$

Use: $\underline{\underline{R}}\underline{\underline{\gamma}} = \frac{1}{2}\underline{\underline{\mathrm{curl}}}_c \underline{\underline{\mathrm{curl}}}_r \underline{\underline{\gamma}} + \frac{1}{2}(\mathrm{div} \underline{M} \underline{\underline{\gamma}})\underline{\underline{\delta}}$,

$\underline{\underline{R}}\underline{\underline{\epsilon}}\underline{\beta} = 0$,

$\underline{\underline{\mathrm{curl}}}_r \underline{\underline{K}} = \underline{\underline{\mathrm{curl}}}_c \underline{\underline{K}} = \underline{\underline{\mathrm{curl}}}_s \underline{\underline{K}}$ if $\underline{M}\underline{\underline{K}} = 0$.

$$
\ddot{\underline{\underline{K}}} = -\underline{\underline{\mathrm{curl}}}_s \underline{\underline{\mathrm{curl}}}_s \underline{\underline{K}} - \underline{\underline{\nabla}}\underline{\nabla}\dot{N}
$$

Introduce $\boxed{\underline{\underline{\nu}} := \dot{\underline{\underline{K}}} \text{ and } \underline{\underline{\mu}} := \underline{\underline{\mathrm{curl}}}_s \underline{\underline{K}}}$

(so $\underline{\underline{K}} = \underline{\underline{K}}(0) + \int_0^t \underline{\underline{\nu}}\, dt$, $\underline{\underline{\gamma}} = \underline{\underline{\gamma}}(0) + \int_0^t (-2\underline{\underline{K}} + 2\underline{\underline{\epsilon}}\underline{\beta})\, dt$).

$$
\begin{aligned}
\dot{\underline{\underline{\nu}}} &= -\underline{\underline{\mathrm{curl}}}_s \underline{\underline{\mu}} - \underline{\underline{\nabla}}\underline{\nabla}\dot{N} \\
\dot{\underline{\underline{\mu}}} &= \underline{\underline{\mathrm{curl}}}_s \underline{\underline{\nu}}
\end{aligned}
$$

← **FOSH**

$$
\underline{\underline{\nu}}(0) = \underline{\underline{R}}\underline{\underline{\gamma}}(0) - \underline{\underline{\nabla}}\underline{\nabla} N(0), \quad \underline{\underline{\mu}}(0) = \underline{\underline{\mathrm{curl}}}_s \underline{\underline{K}}(0).
$$

Constraints and Initial Data

$$\dot{\underline{\underline{\nu}}} = -\underline{\underline{\mathrm{curl}}}_s\,\underline{\underline{\mu}} - \underline{\underline{\nabla}}\,\underline{\nabla}\dot{N}, \quad \dot{\underline{\underline{\mu}}} = \underline{\underline{\mathrm{curl}}}_s\,\underline{\underline{\nu}}$$

$$\underline{\underline{\nu}}(0) = \underline{\underline{R}}\,\underline{\underline{\gamma}}(0) - \underline{\underline{\nabla}}\,\underline{\nabla}N(0), \quad \underline{\underline{\mu}}(0) = \underline{\underline{\mathrm{curl}}}_s\,\underline{\underline{K}}(0)$$

If the initial data $\underline{\underline{\nu}}(0)$, $\underline{\underline{\mu}}(0)$ is derived from ADM initial data which satisfies the Hamiltonian and momentum constraints, then they satisfy the constraints:

$$\underline{M}\,\underline{\underline{\nu}}(0) = \underline{M}\,\underline{\underline{R}}\,\underline{\underline{\gamma}}(0) = -\tfrac{1}{2}\underline{\nabla}\,\mathrm{div}\,\underline{M}\,\underline{\underline{\gamma}}(0) = 0,$$

$$\underline{M}\,\underline{\underline{\mu}}(0) = \underline{M}\,\underline{\underline{\mathrm{curl}}}_s\,\underline{\underline{K}}(0) = \underline{M}\,\underline{\underline{\mathrm{curl}}}_r\,\underline{\underline{K}}(0) = 0.$$

$$\underline{p} := \underline{M}\,\underline{\underline{\nu}}, \ \underline{q} := \underline{M}\,\underline{\underline{\mu}} \implies \boxed{\dot{\underline{p}} = -\tfrac{1}{2}\underline{\mathrm{curl}}\,\underline{q}, \ \dot{\underline{q}} = \tfrac{1}{2}\underline{\mathrm{curl}}\,\underline{p}}$$

Theorem: If the initial data satisfy the constraints $\underline{M}\,\underline{\underline{\nu}}(0) = \underline{M}\,\underline{\underline{\mu}}(0) = 0$, and $\underline{\underline{\nu}}$ and $\underline{\underline{\mu}}$ satisfy the evolution, then the constraints $\underline{M}\,\underline{\underline{\nu}} = \underline{M}\,\underline{\underline{\mu}} = 0$ are satisfied for all time.

Solution Procedure

$$\dot{\underline{\underline{\nu}}} = -\underline{\underline{\mathrm{curl}}}_s\underline{\underline{\mu}} - \underline{\underline{\nabla}}\,\underline{\nabla}\dot{N}$$

$$\dot{\underline{\underline{\mu}}} = \underline{\underline{\mathrm{curl}}}_s\underline{\underline{\nu}}$$

$$\underline{M}\,\underline{\underline{\nu}} = 0$$

$$\underline{M}\,\underline{\underline{\mu}} = 0$$

$$\underline{\underline{\nu}}(0) = \underline{\underline{R}}\,\underline{\underline{\gamma}}(0) - \underline{\underline{\nabla}}\,\underline{\nabla}N(0), \;\; \underline{\underline{\mu}}(0) = \underline{\underline{\mathrm{curl}}}_s\underline{\underline{K}}(0)$$

Theorem: Given initial data $\underline{\underline{\gamma}}(0)$, $\underline{\underline{K}}(0)$ satisfying the constraints,

- define initial data for $\underline{\underline{\nu}}$, $\underline{\underline{\mu}}$
- evolve $\underline{\underline{\nu}}$, $\underline{\underline{\mu}}$
- get $\underline{\underline{K}} = \underline{\underline{K}}(0) + \int_0^t \underline{\underline{\nu}}$, $\underline{\underline{\gamma}} = \underline{\underline{\gamma}}(0) - 2\int_0^t (\underline{\underline{K}} - \underline{\underline{\epsilon}}\,\underline{\beta})$

Then $\underline{\underline{\gamma}}$, $\underline{\underline{K}}$ satisfy the ADM.

On Bounded Domains

Consider now the problem posed on a polyhedron $\Omega \subset \mathbf{R}^3$. Our goal is to find well-posed boundary conditions such that the analog of the previous theorem is true.

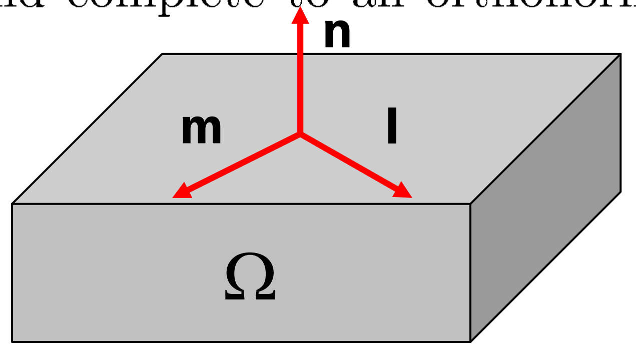
Theorem: Given initial data $\underline{\underline{\gamma}}(0)$, $\underline{\underline{K}}(0)$ on Ω satisfying the constraints,

- define initial data for $\underline{\underline{\nu}}$, $\underline{\underline{\mu}}$
- evolve $\underline{\underline{\nu}}$, $\underline{\underline{\mu}}$ with boundary conditions
- get $\underline{\underline{K}} = \underline{\underline{K}}(0) + \int_0^t \underline{\underline{\nu}}$, $\underline{\underline{\gamma}} = \underline{\underline{\gamma}}(0) - 2\int_0^t (\underline{\underline{K}} - \underline{\underline{\epsilon}}\underline{\beta})$

Then $\underline{\underline{\gamma}}$, $\underline{\underline{K}}$ satisfy the ADM.

Constraint Preserving Boundary Conditions

On any face of Ω, let n be the unit normal, and complete to an orthonormal tetrad n^i, m^i, l^i.



One set: $n^i m^j \nu_{ij} = n^i l^j \nu_{ij} = (m^i m^j - l^i l^j)\mu_{ij} = m^i l^j \mu_{ij} = 0$

or, equivalently,

$$n^i \tau^{jk} \nu_{ij} = 0, \quad (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\mu_{ij} = 0, \quad (\text{ where } \tau^{ij} := \delta^{ij} - n^i n^j)$$

Another set: $n^i m^j \mu_{ij} = n^i l^j \mu_{ij} = (m^i m^j - l^i l^j)\nu_{ij} = m^i l^j \nu_{ij} = 0$

or, equivalently,

$$n^i \tau^{jk} \mu_{ij} = 0, \quad (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\nu_{ij} = 0$$

Strategy

- Write the general form of maximal nonnegative boundary conditions for the system of constraints:

$$\dot{\underline{p}} = -\frac{1}{2}\underline{\operatorname{curl}}\,\underline{q}, \quad \dot{\underline{q}} = \frac{1}{2}\underline{\operatorname{curl}}\,\underline{p}$$

- Trade the transverse derivatives for temporal and tangential ones using the main system.
- Equate the expressions with zero.
- Match these conditions with maximal nonnegative boundary conditions for the main system.

Extended System

$$
\begin{aligned}
\dot{\underline{\underline{\nu}}} &= -\underline{\underline{\mathrm{curl}}}_s \underline{\underline{\mu}} - \underline{\underline{M}}^* \underline{p} - \underline{\underline{\nabla}}\,\underline{\nabla} \dot{N} \\
\dot{\underline{\underline{\mu}}} &= \underline{\underline{\mathrm{curl}}}_s \underline{\underline{\nu}} - \underline{\underline{M}}^* \underline{q} \\
\dot{\underline{a}} &= \underline{M}\,\underline{\underline{\nu}} \\
\dot{\underline{b}} &= \underline{M}\,\underline{\underline{\mu}}
\end{aligned}
$$

FOSH

$$
\underline{\underline{\nu}}(0) = \underline{\underline{R}}\,\underline{\underline{\gamma}}(0) - \underline{\underline{\nabla}}\,\underline{\nabla} N(0), \ \underline{\underline{\mu}}(0) = \underline{\underline{\mathrm{curl}}}_s \underline{\underline{K}}(0), \ \underline{a}(0) = 0, \ \underline{b}(0) = 0
$$

Two sets:

$$
\boxed{n^i \tau^{jk} \nu_{ij} = 0, \ (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\mu_{ij} = 0, \ a_i = 0, \ n^i b_i = 0}
$$

$$
\boxed{n^i \tau^{jk} \mu_{ij} = 0, \ (2\tau^{ik}\tau^{jl} - \tau^{kl}\tau^{ij})\nu_{ij} = 0, \ b_i = 0, \ n^i a_i = 0}
$$

Solution Procedure

Theorem: Given initial data $\underline{\underline{\gamma}}(0)$, $\underline{\underline{K}}(0)$ on Ω satisfying the constraints,

- define initial data for $\underline{\underline{\nu}}$, $\underline{\underline{\mu}}$, and set $\underline{a}(0) = 0$, $\underline{b}(0) = 0$
- evolve $\underline{\underline{\nu}}$, $\underline{\underline{\mu}}$, $\underline{a}$, $\underline{b}$ with boundary conditions

Then $\underline{a} = 0$, $\underline{b} = 0$ for all time, and $\underline{\underline{\gamma}}$, $\underline{\underline{K}}$ satisfy the ADM.

$(\underline{\underline{K}} = \underline{\underline{K}}(0) + \int_0^t \underline{\underline{\nu}},\ \underline{\underline{\gamma}} = \underline{\underline{\gamma}}(0) - 2\int_0^t (\underline{\underline{K}} - \underline{\underline{\epsilon}}\underline{\beta}))$

Thank You!