On strong hyperbolicity of Einstein's equations

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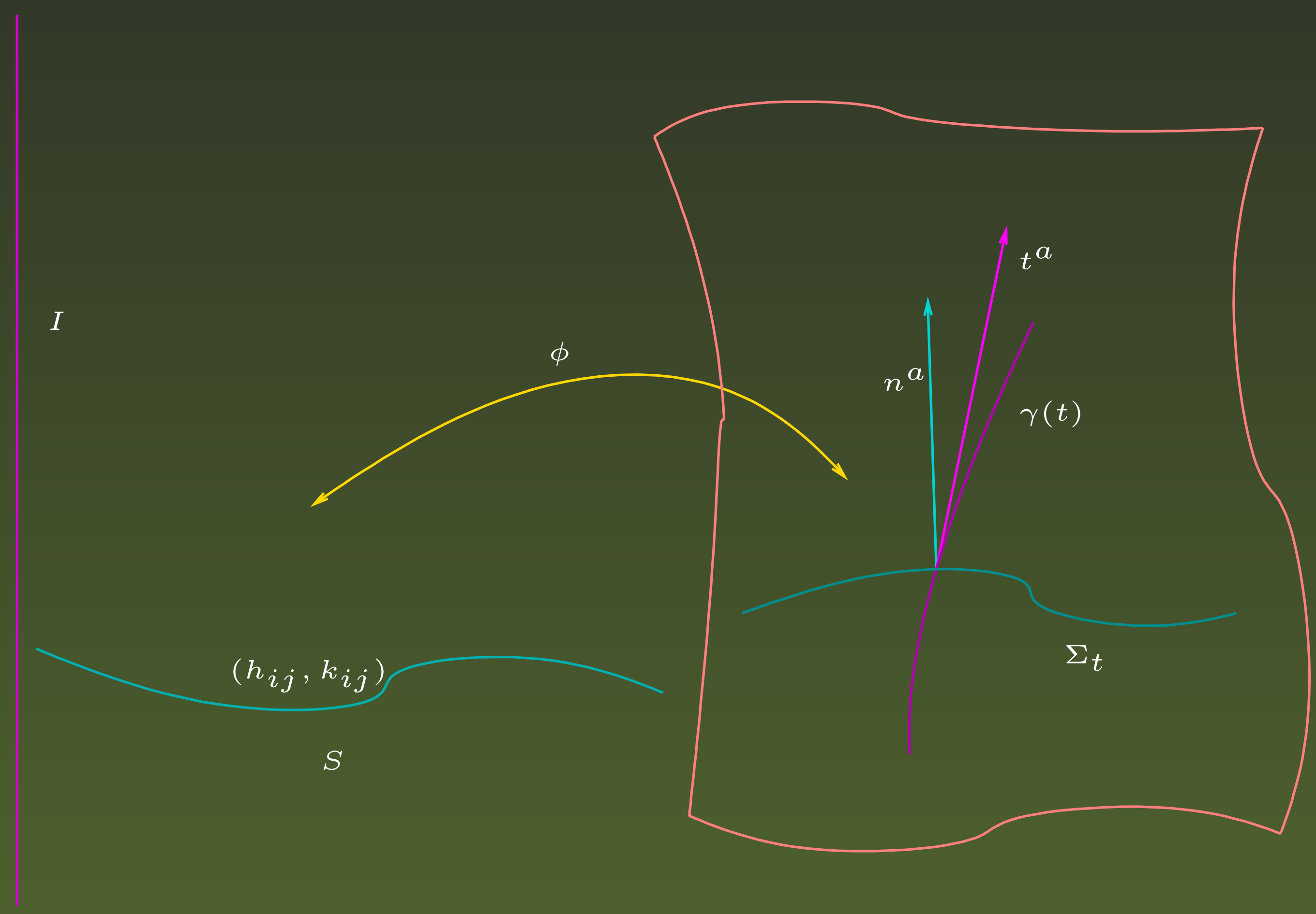
OUTLINE

- Introduction
  - The setting
  - Questions
  - Math
  - More questions
  - Some answers
- Applications
  - ADM-BSSN hyperbolicity
  - Subsidiary system hyperbolicity

The Setting



ADM equations I

$$G_{ab} = 0 \quad \Rightarrow \left\{ \begin{array}{c} \partial_t h_{ij} = -2NK_{ij} + \nabla_{(i}N_{j)}, \\ \\ \partial_t K_{ij} = -\nabla_i \nabla_j N + N[{}^3R_{ij}(h) + K_{ij}(\mathrm{tr}K) - 2K_{im}K_j{}^m] \\ +[N^m \nabla_m K_{ij} + \nabla_i N^m K_{jm} + \nabla_j N^m K_{im}], \\ \\ {}^3R(h) + \mathrm{tr}K^2 - k_{ij}k^{ij} = 0, \\ \\ \nabla^j K_{ij} - D_i \mathrm{tr}K = 0, \end{array} \right.$$

$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dx^i \otimes dt + dt \otimes dx^i) + h_{ij} dx^i \otimes dx^j$$

ADM equations II

$$G_{ab} = 0 \quad \Rightarrow \left\{ \begin{array}{c} \partial_t h_{ij} = -2NK_{ij} + \nabla_{(i}N_{j)}, \\ \\ \partial_t K_{ij} = -\nabla_i \nabla_j N + N[{}^3R_{ij}(h) + K_{ij}(\mathrm{tr}K) - 2K_{im}K_j{}^m] \\ +[N^m \nabla_m K_{ij} + \nabla_i N^m K_{jm} + \nabla_j N^m K_{im}], \\ \\ {}^3R(h) + \mathrm{tr}K^2 - k_{ij}k^{ij} = 0, \\ \\ \nabla^j K_{ij} - D_i \mathrm{tr}K = 0, \end{array} \right.$$

$$g = -(N^2 - N_i N^i) dt \otimes dt + N_i (dx^i \otimes dt + dt \otimes dx^i) + h_{ij} dx^i \otimes dx^j$$

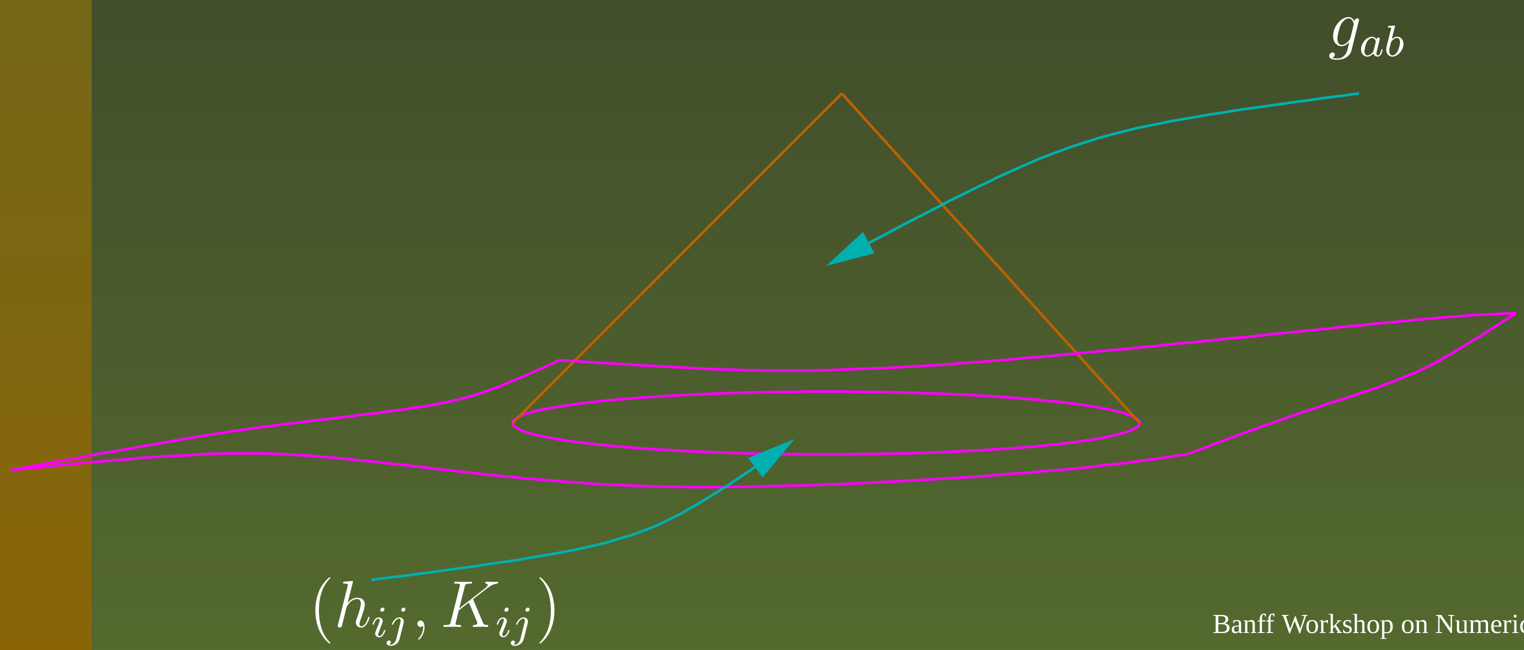
Einstein's equations

Questions (Mostly answered by I. Choquet-Bruhat 50 years ago using another system of equations):

- Given smooth initial data (h_{ij}, K_{ij}) is there a unique solution?
- Causality?
- Are the evolution equations unique?
- Which evolutions are stable? (well posed)
- Are the constraints satisfied along time evolution if they are satisfied initially?
- Which evolution equations satisfy the constraint quantities? (Subsidiary systems)

Einstein's equations

- Given smooth initial data (h_{ij}, K_{ij}) is there a unique solution?
  - No: in terms of metrics: the diffeomorphism freedom implies that if g_{ab} is a solution to Einstein's equations, also is $\phi_\star g_{ab}$ where ϕ is any smooth diffeomorphism.
  - Yes: in terms of geometries, the equivalent class of metrics under diffeomorphisms.



Einstein’s equations

- Are the evolution equations unique?
  - No: you can add constraint equations to the evolution equations
  - No: you can fix some algebraic or differential relation between some components (exploit the gauge freedom)

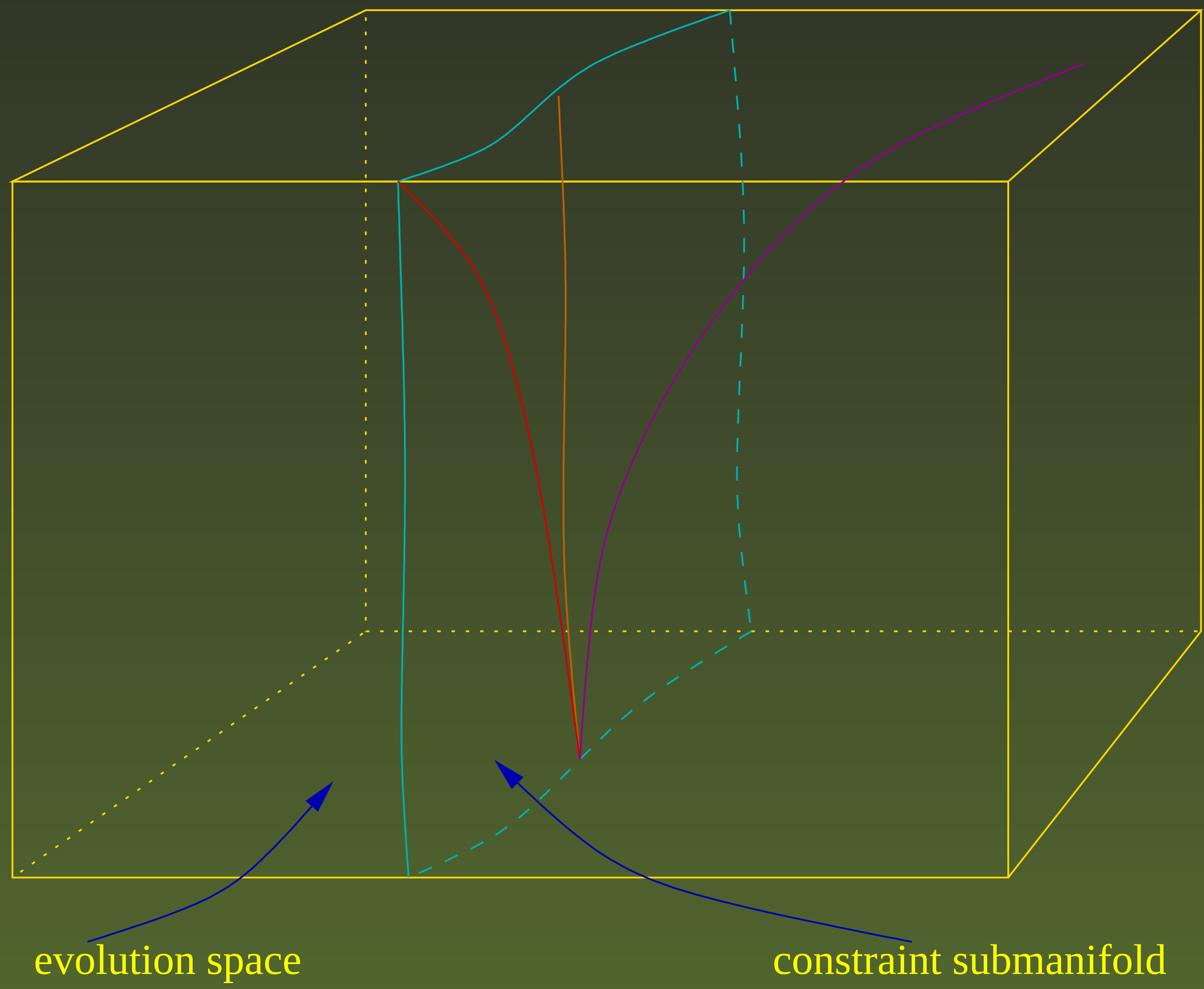
Einstein's equations

- Which evolutions are stable? (well posed)
  - We know that many systems are well posed and others not, notably the ADM system is only weakly hyperbolic.
  - There are many (several parameter families) that are symmetric hyperbolic.
  - There are some which are only strongly hyperbolic
  - There are second orther systems which are well posed (for can be reduced to first order pseudodifferential systems)
  - These systems are all linearly degenerated, so, unlike fluids, no shocks seems to form.
  - Numerically solutions can behave very bad but even when the system is well posed.

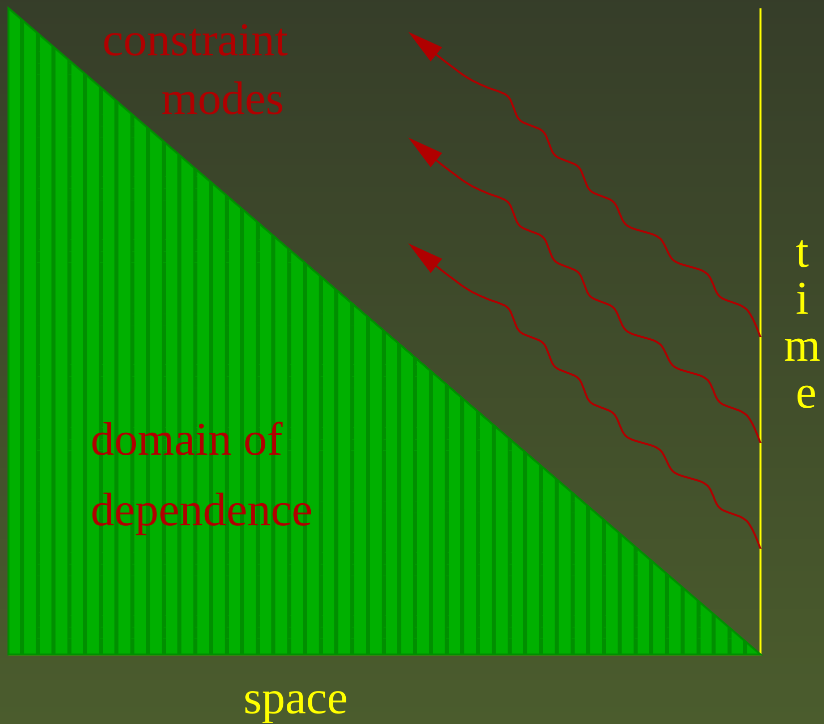
Einstein's equations

- Are the constraints satisfied along time evolution if they are satisfied initially?
  - Yes: at the abstract level and for the initial value problem.
  - For each particular system one has to check that the subsidiary system has a unique (zero) solution.
  - Numerically there can be *constraint modes* which diverge exponentially from the constraint surface.
  - In many numerical problems one has a initial-boundary value problem, very little is known in this case, for constraint preserving boundary conditions must be given and in general they do not result in a well posed system. More in Sarbach's talk.

Constraint Propagation



Constraint Propagation



Constraint Propagation

- Which evolution equations satisfy the constraint quantities? (Subsidiary systems)
  - Provided the evolution system is strongly hyperbolic, and that the constraint satisfy certain condition, then the subsidiary system they satisfy is also strongly hyperbolic.
  - Furthermore the characteristics of the subsidiary system are a subset of the characteristics of the evolution system.
  - There exist symmetric hyperbolic systems whose constraint propagation is not symmetric hyperbolic (but strongly hyperbolic).

Applications:

- ADM-BSSN first-second order systems [Frittelli-R, Sarbach-Calabrese-Pullin-Tiglio, Kreiss-Ortiz, Nagy-Ortiz-R]
- Constraint propagation. [Hyperbolicity properties of subsidiary systems of constraints.]

ADM equations I

$$G_{ab} = 0 \quad \Rightarrow \left\{ \begin{array}{c} \mathcal{L}_n h_{ab} = -2k_{ab}, \\ \\ \mathcal{L}_n k_{ab} = {}^{(3)}R_{ab} - 2k_a{}^c k_{bc} + k_{ab} k_c{}^c - \frac{D_a D_b N}{N}, \\ \\ {}^{(3)}R + (k_c{}^c)^2 - k_{ab} k^{ab} = 0, \\ \\ D^b k_{ba} - D_a k = 0, \end{array} \right.$$

ADM equations II

$$
\begin{aligned}
\mathcal{L}_{(t-\beta)} h_{ij} &= -2N k_{ij} \\
\mathcal{L}_{(t-\beta)} k_{ij} &= \frac{N}{2} h^{kl} \left[-\partial_k \partial_l h_{ij} - \partial_i \partial_j h_{kl} + 2\partial_k \partial_{(i} h_{j)l} \right] + B_{ij}
\end{aligned}
$$

where

$$
\begin{aligned}
B_{ij} &:= N \left[\gamma_{ikl} \gamma_j{}^{kl} - \gamma_{ij}{}^k \gamma_{kl}{}^l - 2k_i{}^l k_{jl} + k_{ij} k_l{}^l - A_{ij} \right], \\
\gamma_{ij}{}^k &:= \frac{1}{2} h^{kl} (2\partial_{(i} h_{j)k} - \partial_k h_{ij}), \\
A_{ij} &:= a_i a_j - \gamma_{ij}{}^k a_k - 2\gamma_{ikl} \gamma_j{}^{(kl)} + \partial_i [(\partial_j N)/N], \\
a_i &:= (\partial_i N)/N.
\end{aligned}
$$

ADM equations III

Hyperbolicity analysis: 1) consider only the principal part, 2) freeze coefficients, 3) substitute all derivatives by Fourier transforms ($\partial_k h_{ij} \to i\omega_k \hat{h}_{ik}$), and 4) define $\hat{\ell}_{ij} = i\omega \hat{h}_{ij}$. [Kreiss, Ortiz][Taylor]

The associated first order system is then

$$\partial_t \hat{\ell}_{ij} \quad \hat{=} \quad i\omega \left[-2N\hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right],$$

$$\partial_t \hat{k}_{ij} \quad \hat{=} \quad i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + \tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]$$

with $\tilde{\omega}_i = \omega_i / \omega$.

Result:

- ADM equations are only weakly hyperbolic (3 eigenvectors missing).

ADM equations IV

$N = h^b Q$ (h = determinant of h_{ij})

The associated first order system is then

$$\partial_t \hat{\ell}_{ij} \hat{=} i\omega \left[-2N\hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij} \right],$$

$$\partial_t \hat{k}_{ij} \hat{=} i\omega \left[-\frac{N}{2} \left(\hat{\ell}_{ij} + (1+b)\tilde{\omega}_i \tilde{\omega}_j h^{kl} \hat{\ell}_{kl} - 2\tilde{\omega}^k \tilde{\omega}_{(i} \hat{\ell}_{j)k} \right) + \tilde{\omega}_k \beta^k \hat{k}_{ij} \right]$$

Result:

- Modified ADM equations for $b > 0$ still weakly hyperbolic (2 eigenvectors missing).
- Adding Hamiltonian constraint does not change hyperbolicity, but does change characteristics.

BSSN equations I

$$f^k = h^{ij}\gamma_{ij}{}^k + dh^{kl}\gamma_{lm}{}^m = h^{kl}(h^{ij}\partial_i h_{jl} + \partial_l \ln h)$$

$$\begin{aligned}
\mathcal{L}_{(t-\beta)} h_{ij} &= -2Nk_{ij} \\
\mathcal{L}_{(t-\beta)} k_{ij} &= \frac{N}{2} h^{kl}\left[-\partial_k\partial_l h_{ij} - b\,\partial_i\partial_j h_{kl}\right] + N\partial_{(i} f_{j)} + \mathcal{B}_{ij} \\
\mathcal{L}_{(t-\beta)} f_i &= N[-(2-c)D^k k_{ki} + (1-c)D_i k_k{}^k] + \mathcal{C}_i
\end{aligned}$$

BSSN equations II

Hyperbolicity Analysis:

$$\begin{aligned}
\partial_t \hat{\ell}_{ij} &\hat{=} i\omega \left[-2\alpha \hat{k}_{ij} + \tilde{\omega}_k \beta^k \hat{\ell}_{ij}\right] \\
\partial_t \hat{k}_{ij} &\hat{=} i\omega \left[\frac{\alpha}{2}\left(-\hat{\ell}_{ij} - b\,\tilde{\omega}_i\tilde{\omega}_j h^{kl}\hat{\ell}_{kl} + 2\tilde{\omega}_{(i}\hat{f}_{j)}\right) + \tilde{\omega}_k \beta^k \hat{k}_{ij}\right] \\
\partial_t \hat{f}_i &\hat{=} i\omega \left[\alpha\left((-2+c)\hat{k}_{ik}\tilde{\omega}^k + (1-c)\tilde{\omega}_i h^{kl}\hat{k}_{kl}\right) + \tilde{\omega}_k \beta^k \hat{f}_i\right]
\end{aligned}$$

Result: [Nagy-Ortiz-R]

- Modified BSSN equations for $b > 0$ $c > 0$ strongly hyperbolic.
- Eigenvalues: $(0, \pm 1, \pm\sqrt{b}, \pm\sqrt{c/2})$

Constraint Propagation

Evolution System:

$$\partial_t u^\alpha = A(u,t,x)^{\alpha a}{}_\beta \partial_a u^\beta + B(u,t,x)^\alpha,$$

Constraints:

$$C^A = K(u,t,x)^{Aa}{}_\beta \partial_a u^\beta + L(u,t,x)^A,$$

Integrability condition (subsidiary system):

$$\partial_t C^A = S(u,t,x)^{Aa}{}_B \partial_a C^B + R(u,\partial u,t,x)^A{}_B C^B,$$

- Want to study what can we say about the properties of the subsidiary system from what we know from the evolution system.

Constraint Propagation II

- Problem: In general $S(u,t,x)^{Aa}{}_B$ is not unique if the constraint themselves satisfy certain identities.

For instance, if there is an $L_A(\omega)$ such that:

$$L_A(\omega)K^{An}{}_\alpha \omega_n = 0$$

we could add to $S(u,t,x)^{Aa}{}_B$

$$M^{Aa}L_B$$

- With this addition there are easy examples where one can get any sort of badly posed systems!

Constraint Propagation III

- Assume: For any ω_i, $K^{An}{}_{\alpha}\omega_n$ is surjective.
- In general this is not satisfied, but in examples of interest one finds subset of constraints which do satisfy it. [Maxwell, EC].

Constraint Propagation IV

Integrability condition implies:

$$K^{A(a}{}_{\alpha}A^{|\alpha|b)}{}_{\beta} - S^{A(a}{}_{B}K^{|B|b)}{}_{\beta} = 0$$

- **Lemma 1:** Given any fixed non-vanishing co-vector ω_a. If (σ, u^α) is an eigenvalue-eigenvector pair of $A^{\alpha a}{}_{\beta}\omega_a$ then $(\sigma, v^A = K^{Aa}{}_{\alpha}\omega_a u^\alpha)$, if v^A is non-vanishing, is an eigenvalue-eigenvector pair of $S^{Aa}{}_{B}\omega_a$.

Constraint Propagation IV

Integrability condition implies:

$$[K^{Aa}{}_{\alpha}\omega_a A^{\alpha b}{}_{\beta}\omega_b - S^{Aa}{}_{B}\omega_a K^{Bb}{}_{\beta}\omega_b]u^{\beta} = 0$$

- **Lemma 1:** Given any fixed non-vanishing co-vector ω_a. If (σ, u^{α}) is an eigenvalue-eigenvector pair of $A^{\alpha a}{}_{\beta}\omega_a$ then $(\sigma, v^A = K^{Aa}{}_{\alpha}\omega_a u^{\alpha})$, if v^A is non-vanishing, is an eigenvalue-eigenvector pair of $S^{Aa}{}_{B}\omega_a$.

Constraint Propagation IV

Integrability condition implies:

$$K^{Aa}{}_{\alpha}\omega_a A^{\alpha b}{}_{\beta}\omega_b u^{\beta} - S^{Aa}{}_{B}\omega_a v^{B} = 0$$

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Constraint Propagation IV

Integrability condition implies:

$$K^{Aa}{}_{\alpha}\omega_a \sigma u^{\alpha} - S^{Aa}{}_{B}\omega_a v^{B} = 0$$

- **Lemma 1:** Given any fixed non-vanishing co-vector ω_a. If (σ, u^{α}) is an eigenvalue-eigenvector pair of $A^{\alpha a}{}_{\beta}\omega_a$ then $(\sigma, v^A = K^{Aa}{}_{\alpha}\omega_a u^{\alpha})$, if v^A is non-vanishing, is an eigenvalue-eigenvector pair of $S^{Aa}{}_{B}\omega_a$.

Constraint Propagation IV

Integrability condition implies:

$$\sigma v^A - S^{Aa}{}_B \omega_a v^B = 0$$

- **Lemma 1:** Given any fixed non-vanishing co-vector ω_a. If (σ, u^α) is an eigenvalue-eigenvector pair of $A^{\alpha a}{}_\beta \omega_a$ then $(\sigma, v^A = K^{Aa}{}_\alpha \omega_a u^\alpha)$, if v^A is non-vanishing, is an eigenvalue-eigenvector pair of $S^{Aa}{}_B \omega_a$.
- **Corollary 1:** If the evolution system is strongly hyperbolic then so is the subsidiary system. [It does not work symmetric $\longrightarrow$ symmetric].
- **Corollary 2:** The characteristics of the subsidiary system are a subset of the characteristics of the evolution system. The domain of dependence of the subsidiary system is at least as large as the domain of dependence of the evolution system.