Toward Binary Black Hole Simulations in Numerical Relativity

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Outline

- generalized harmonic coordinates
  - definition & utility in GR
  - a numerical evolution scheme based on this form of the field equations
  - choosing the slicing/spatial gauge
  - constraint damping
  - some details of the numerical code
- early simulation results
  - merger of an eccentric black hole binary

Numerical relativity using generalized harmonic coordinates – a brief overview

- Formalism
  - the Einstein equations are re-expressed in terms of generalized harmonic coordinates
    - add *source functions* to the definition of harmonic coordinates to be able to choose arbitrary slicing/gauge conditions
  - add constraint damping terms to aid in the stable evolution of black hole spacetimes
- Numerical method
  - equations discretized using finite difference methods
  - *directly* discretize the metric; i.e. not reduced to first order form
  - use adaptive mesh refinement (AMR) to adequately resolve all relevant spatial/temporal length scales (still need supercomputers in 3D)
  - use (dynamical) excision to deal with geometric singularities that occur inside of black holes
  - add numerical dissipation to eliminate high-frequency instabilities that otherwise tend to occur near black holes
  - use a coordinate system compactified to spatial infinity to place the physically correct outer boundary conditions

Generalized Harmonic Coordinates

- Generalized harmonic coordinates introduce a set of arbitrary *source functions* H^u into the usual definition of harmonic coordinates

$$\nabla^{\alpha}\nabla_{\alpha}x^{\mu} \equiv \frac{1}{\sqrt{-g}}\partial_{\alpha}\left(\sqrt{-g}\,g^{\alpha\mu}\right) = H^{\mu}$$

- When this condition (specifically its gradient) is substituted for certain terms in the Einstein equations, *and the H^u are promoted to the status of independent functions*, the principle part of the equation for *each* metric element reduces to a simple wave equation

$$g^{\gamma\delta}g_{\alpha\beta,\gamma\delta} + \ldots = 0$$

Generalized Harmonic Coordinates

- The claim then is that a solution to the coupled Einstein-harmonic equations

$$g^{\gamma\delta} g_{\alpha\beta,\gamma\delta} + 2 g^{\gamma\delta}{}_{,(\alpha} g_{\beta)\delta,\gamma} + 2H_{(\alpha,\beta)} - 2H_{\delta}\Gamma^{\delta}_{\alpha\beta} + 2\Gamma^{\gamma}_{\delta\beta}\Gamma^{\delta}_{\gamma\alpha} + 8\pi\left(2T_{\alpha\beta} - g_{\alpha\beta}T\right) = 0$$

which include (arbitrary) evolution equations for the source functions, plus addition matter evolution equations, will also be a solution to the Einstein equations *provided* the harmonic constraints

$$C^{\mu} \equiv H^{\mu} - \nabla^{\alpha}\nabla_{\alpha} x^{\mu}$$

and their first time derivative are satisfied at the initial time.

- "Proof" $\nabla^{\alpha}\nabla_{\alpha} C^{\mu} = -R^{\mu}{}_{\nu} C^{\mu}$

An evolution scheme based upon this decomposition

- The idea (following Garfinkle *[PRD 65, 044029 (2002)]*; see also Szilagyi & Winicour *[PRD 68, 041501 (2003)]*) is to construct an evolution scheme based directly upon the preceding equations
  - one can view the source functions as being analogous to the lapse and shift in an ADM style decomposition, encoding the 4 coordinate degrees of freedom
  - the system of equations is manifestly hyperbolic (if the metric is non-singular and maintains a definite signature)
  - the ”constraint” equations are the generalized harmonic coordinate conditions

A 3D numerical code based upon this scheme

- Attractive features for a numerical code
  - wave nature of each equation suggests that it will be straight-forward to discretize using standard AMR techniques developed for hyperbolic equations
  - the fact that the principle part of each equation is a wave equation suggests a simple, *direct* discretization scheme (leapfrog) :
    - no first order quantities are introduced, i.e. the fundamental discrete variables are the metric elements
    - the resulting system of equations has the minimal number of constraints possible (4) for a general, Cauchy-based Einstein gravity code
      - simpler to control “constraint violating modes” when present
- an additional numerical issue we wanted to explore with this code is the use of a spatially compactified coordinate system to apply correct asymptotically flat boundary conditions

Coordinate Issues

- The source functions encode the coordinate degrees of freedom of the spacetime
  - how does one specify H^u to achieve a particular slicing/spatial gauge?
  - what class of evolutions equations for H^u can be used that will not adversely affect the well posedness of the system of equations?

Specifying the spacetime coordinates

- A way to gain insight into how a given H^u could affect the coordinates is to appeal to the ADM metric decomposition

$$ds^2 = -\alpha^2 dt^2 + h_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

then

$$H \cdot n \equiv H_\mu n^\mu = -n^\mu \partial_\mu \ln\alpha - K$$

$$\perp H^i \equiv H_\mu h^{i\mu} = \frac{1}{\alpha} n^\mu \partial_\mu \beta^i + h^{ij}\partial_j \ln\alpha - \overline{\Gamma}^i{}_{jk} h^{jk}$$

or

$$\partial_t \alpha = -\alpha^2 H \cdot n + \ldots$$

$$\partial_t \beta^i = \alpha^2 \perp H^i + \ldots$$

Specifying the spacetime coordinates

- Therefore, H^t (H^i) can be chosen to *drive* α (β^i) to desired values
  - for example, the following slicing conditions are all designed to keep the lapse from “collapsing”, and have so far proven useful in removing some of the coordinate problems with harmonic time slicing

$$H_t = \xi \frac{\alpha - 1}{\alpha^n}$$

$$\partial_t H_t = \xi \partial_t \left(\frac{\alpha - 1}{\alpha^n} \right)$$

$$\nabla^\mu \nabla_\mu H_t = -\xi \frac{\alpha - 1}{\alpha^n} - \zeta \partial_t H_t$$

Constraint Damping

- Following a suggestion by C. Gundlach (based on earlier work by Brodbeck et al *[J. Math. Phys. 40, 909 (1999)]*) modify the Einstein equations in harmonic form as follows:

$$g^{\alpha\beta} g_{\mu\nu,\alpha\beta} + \ldots + \kappa\left(n_\mu C_\nu + n_\nu C_\mu - g_{\mu\nu} n^\alpha C_\alpha\right) = 0$$

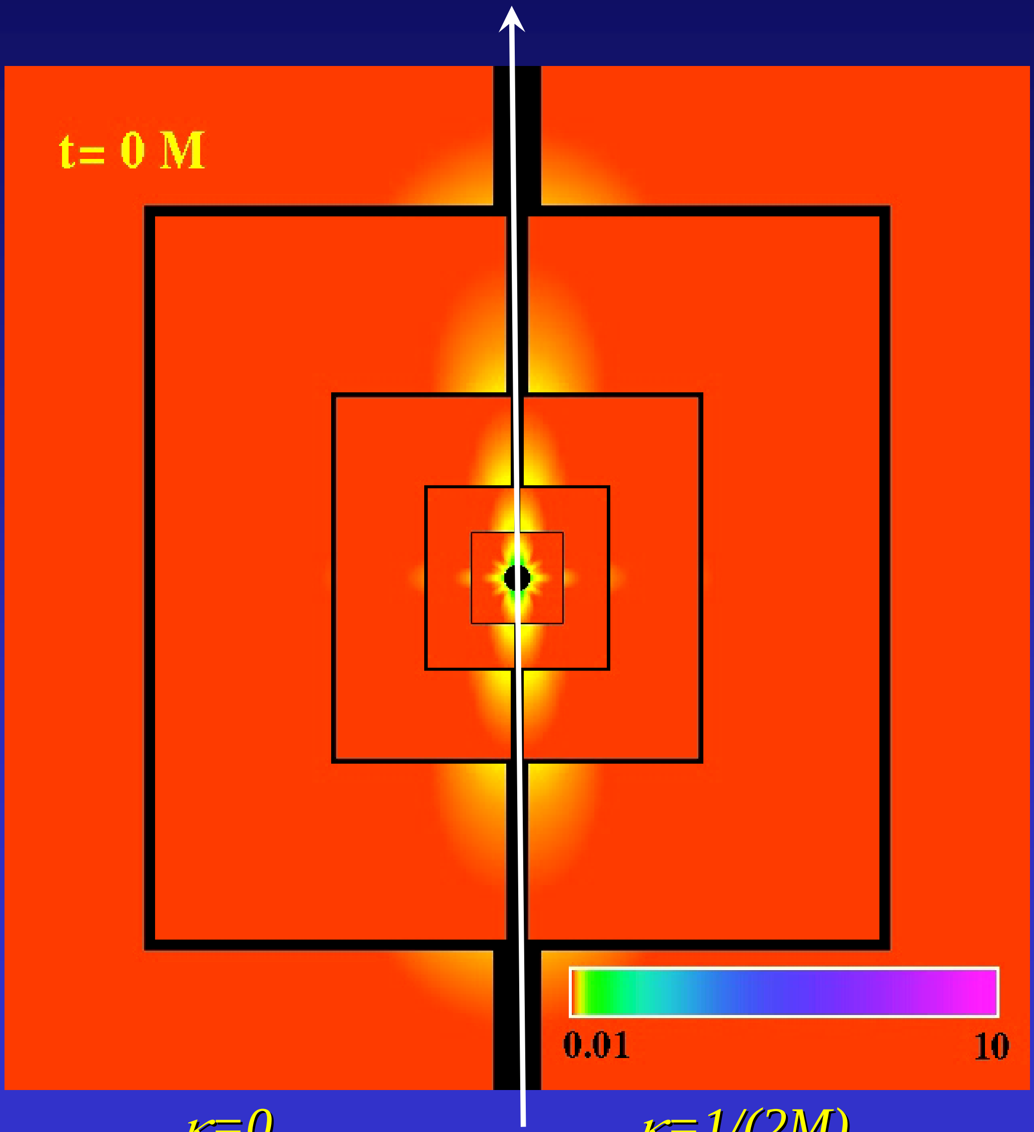
where

$$C^\mu \equiv H^\mu - \nabla^\alpha \nabla_\alpha x^\mu$$

$$n_\mu \equiv -\alpha \nabla_\mu t$$

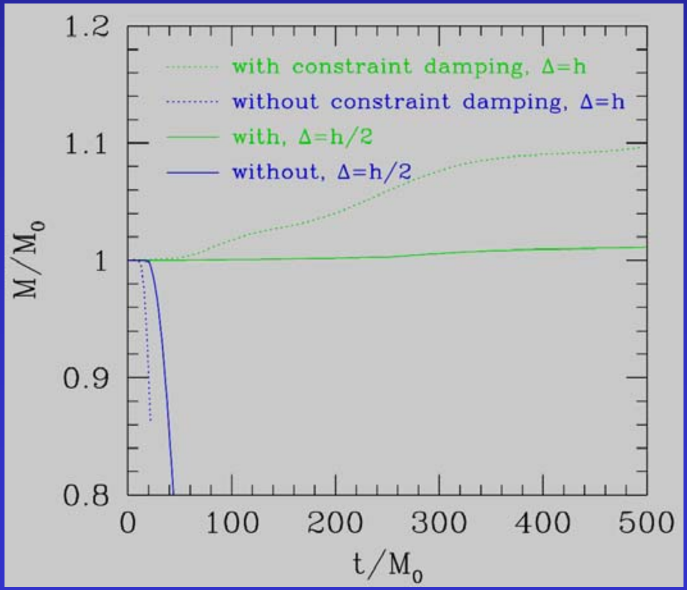
- For positive κ, Gundlach et al have shown that all constraint-violations with finite wavelength are damped for linear perturbations around flat spacetime

Effect of constraint damping



$\kappa=0$ $\kappa=1/(2M)$

- Axisymmetric simulation of a Schwarzschild black hole
- Left and right simulations use *identical* parameters except for the use of constraint damping



An early result – merger of an eccentric binary system

- Initial data
  - at this stage I am most interested in the dynamics of binary systems in general relativity, and not with trying to produce an initial set-up that mimics a particular astrophysical scenario
  - hence, use *boosted scalar field collapse* to set up the binary
  - choice for initial geometry and scalar field profile:
    - spatial metric and its first time derivative is conformally flat
    - maximal (gives initial value of lapse and time derivative of conformal factor) and harmonic (gives initial time derivatives of lapse and shift)
    - Hamiltonian and Momentum constraints solved for initial values of the conformal factor and shift, respectively
  - advantages of this approach
    - "simple" in that initial time slice is singularity free
    - all non-trivial initial geometry is driven by the scalar field—when the scalar field amplitude is zero we recover Minkowski spacetime
  - disadvantages
    - ad-hoc in choice of parameters to produce a desired binary system
    - uncontrollable amount of "junk" initial radiation (scalar and gravitational) in the spacetime; though *all* present initial data schemes suffer from this

An early result – merger of an eccentric binary system

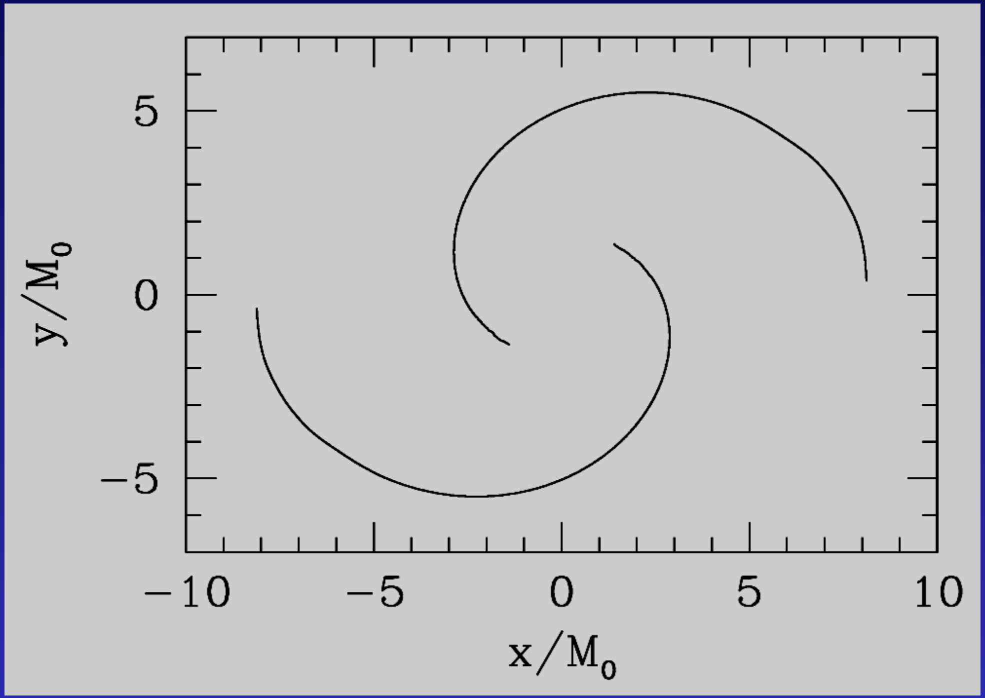
- Gauge conditions:

$$\nabla^{\mu}\nabla_{\mu}H_t = -\xi\frac{\alpha-1}{\alpha^n} - \zeta\partial_t H_t,$$
$$H_i = 0$$
$$\xi \sim 6/M, \zeta \sim 1/M, n = 5$$

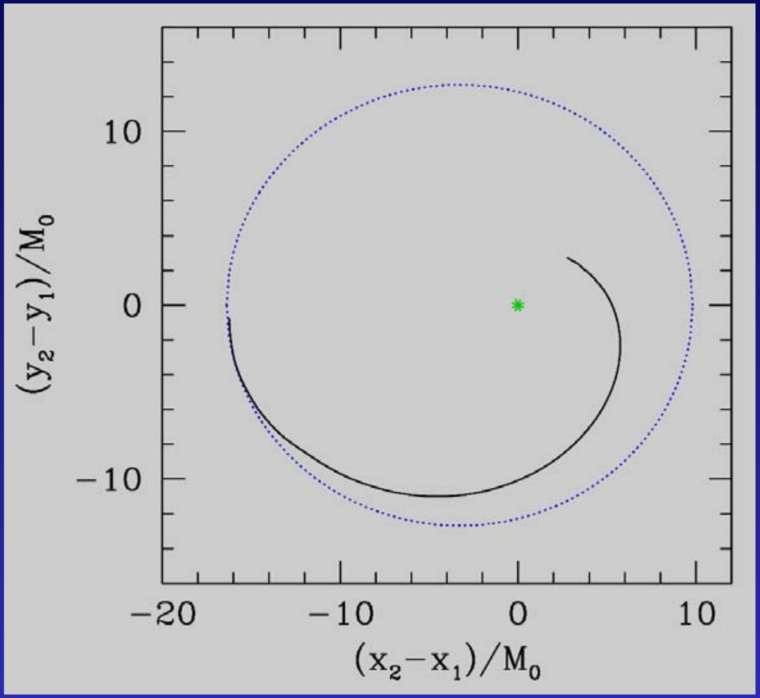
  - Note: this is strictly speaking *not* spatial harmonic gauge, which is defined in terms of the "vector" components of the source function

- Constraint damping term $\kappa \sim 1/M$

Orbit



Simulation (center of mass) coordinates

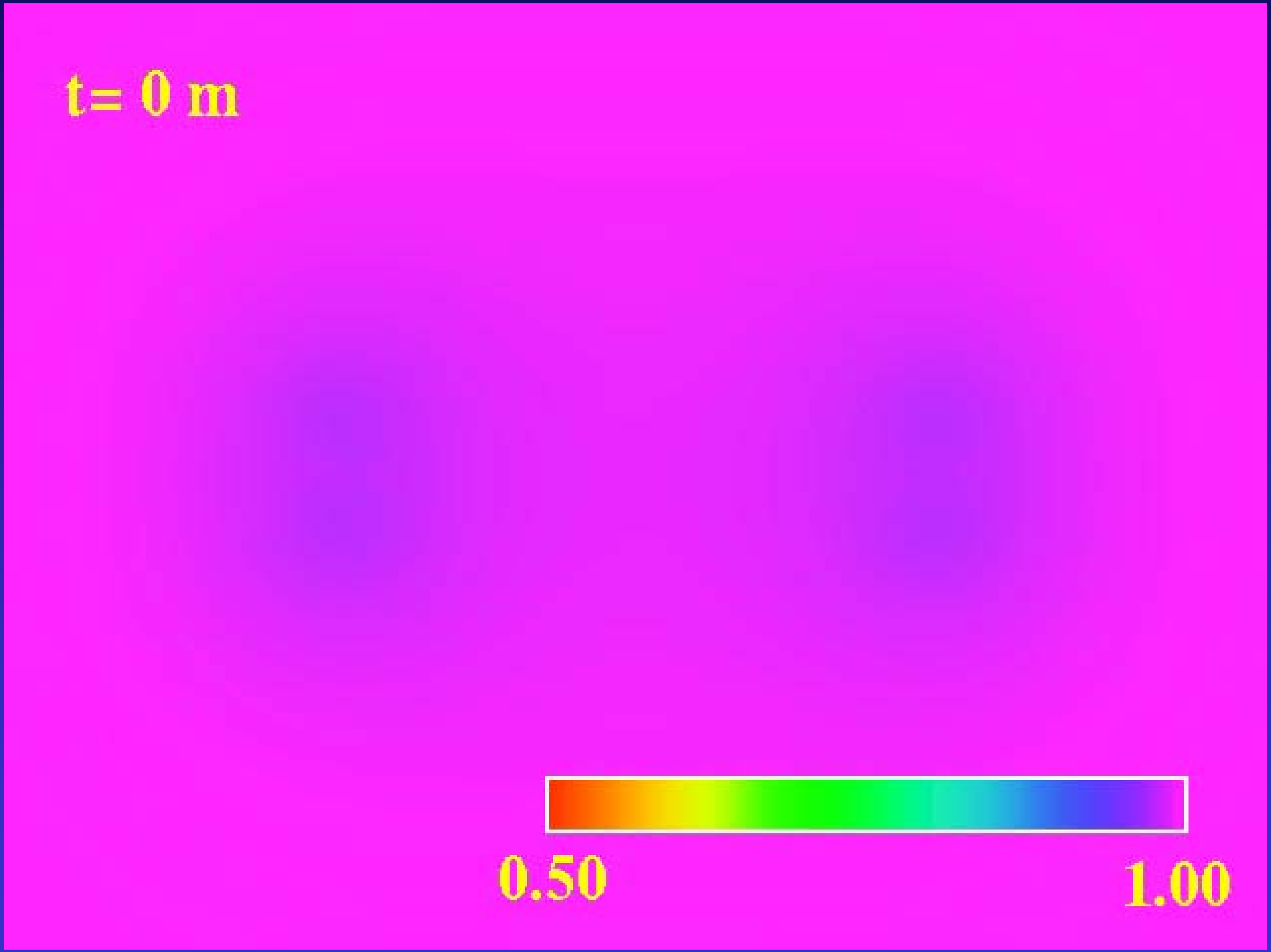


Reduced mass frame; solid black line is position of BH 1 relative to BH 2 (green star); dashed blue line is reference ellipse

- Initially:
  - equal mass components
  - eccentricity e ~ *0.25*
  - coordinate separation of black holes ~ 16M
  - proper distance between horizons ~ 20M
  - velocity of each black hole ~0.12
  - spin angular momentum = 0

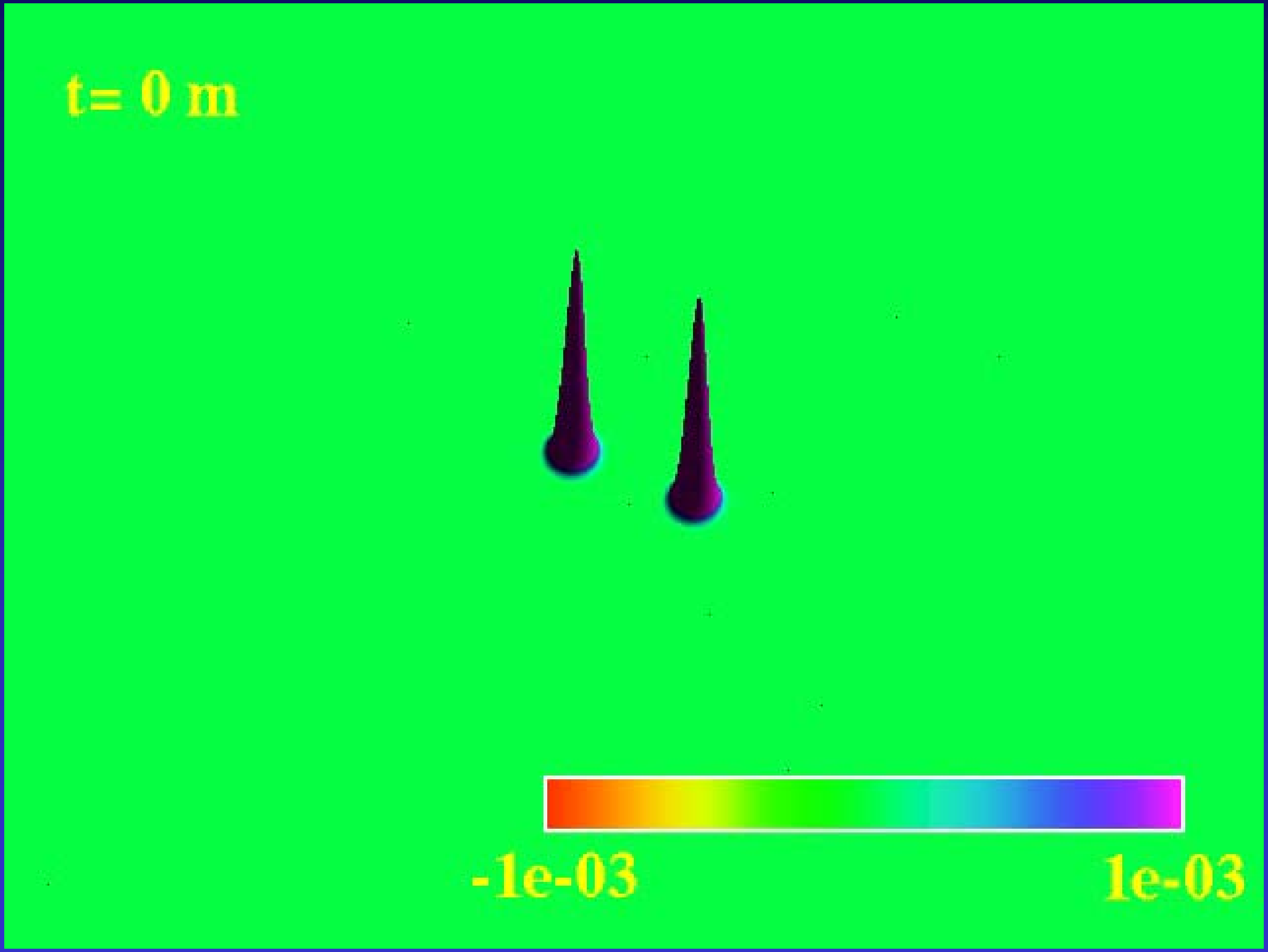
- Final black hole:
  - M_f ~ 1.85M
  - Kerr parameter a ~ 0.7
  - error ~10% ??

Lapse function α



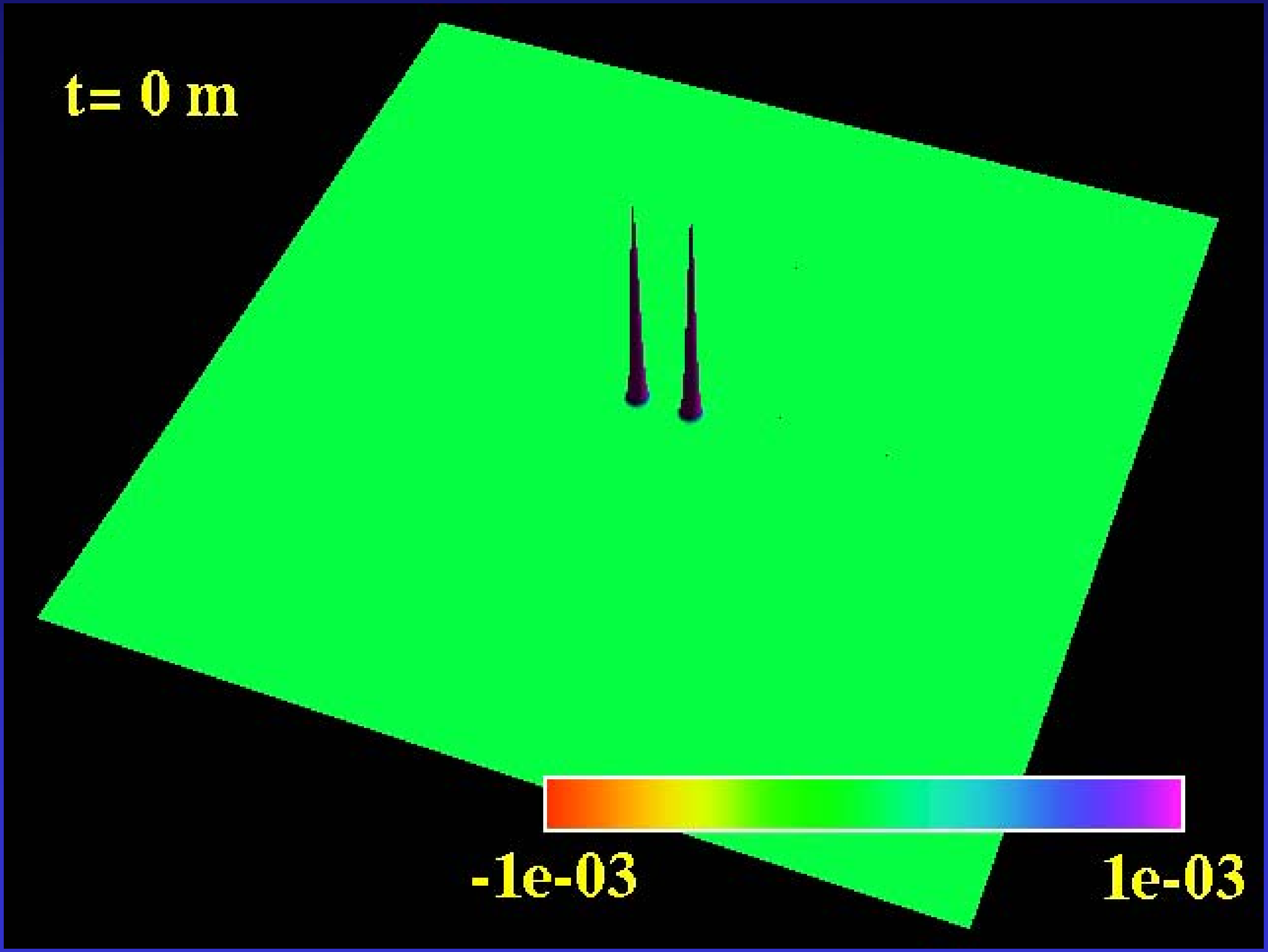
All animations: z=0 slice, time in units of the mass of a single, initial black hole

Scalar field $\phi.r$, uncompactified coordinates

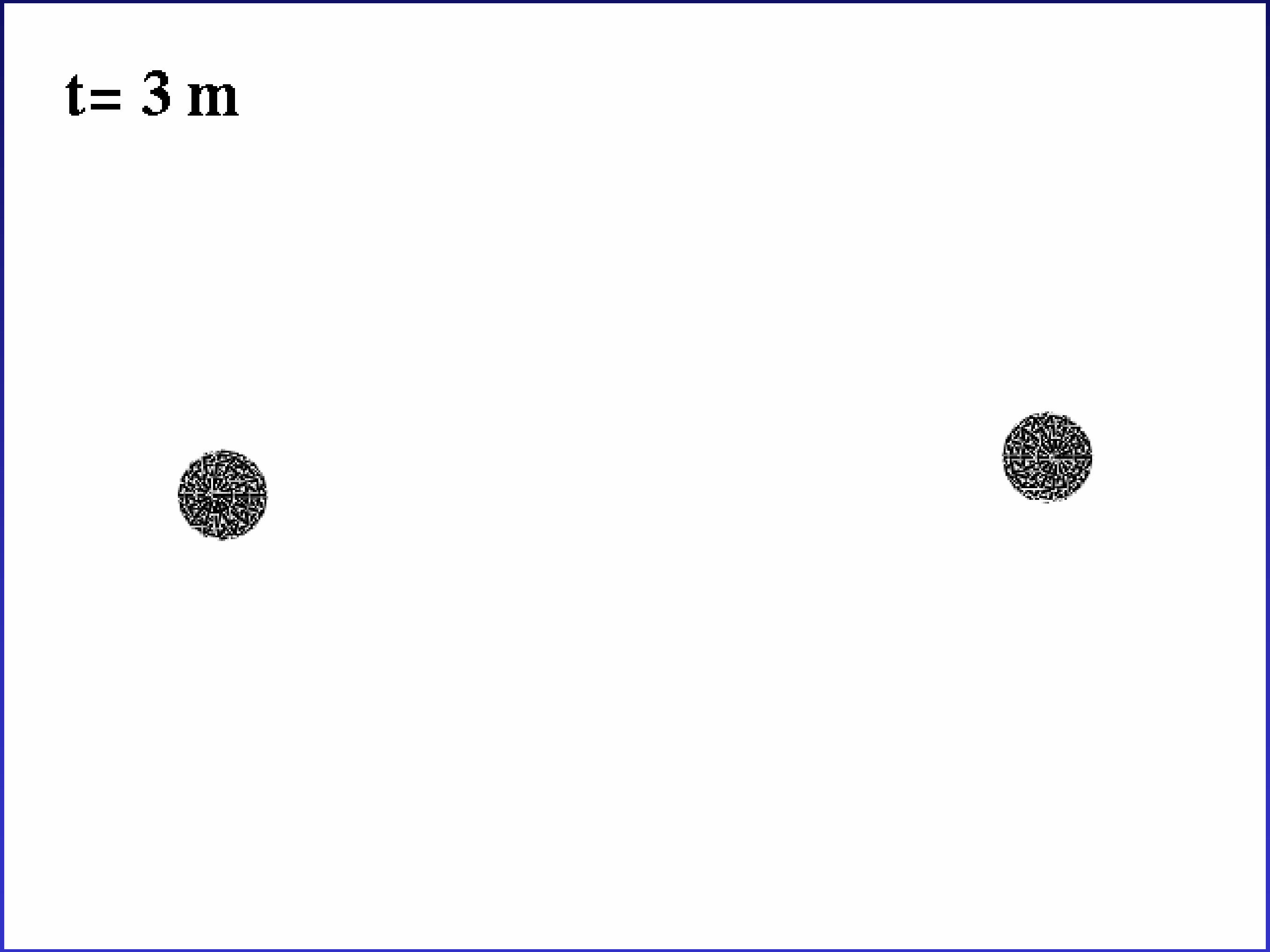


Scalar field $\phi.r$, compactified (code) coordinates

$$\bar{x} = \tan(x\pi/2), \bar{y} = \tan(y\pi/2), \bar{z} = \tan(z\pi/2)$$

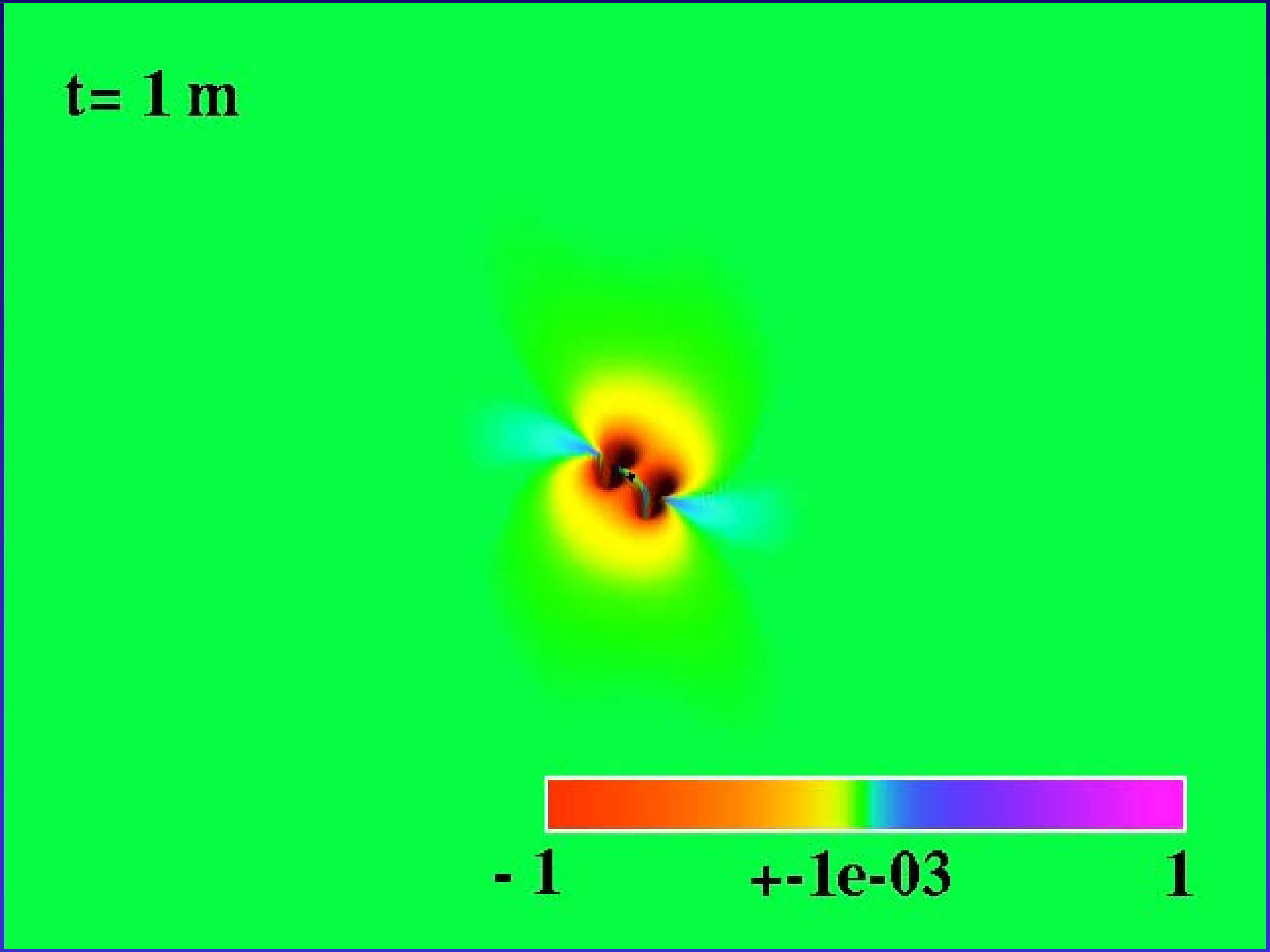


Apparent horizons



Coordinate shape of apparent horizons, viewed from directly above the orbital plane

Gravitational waves

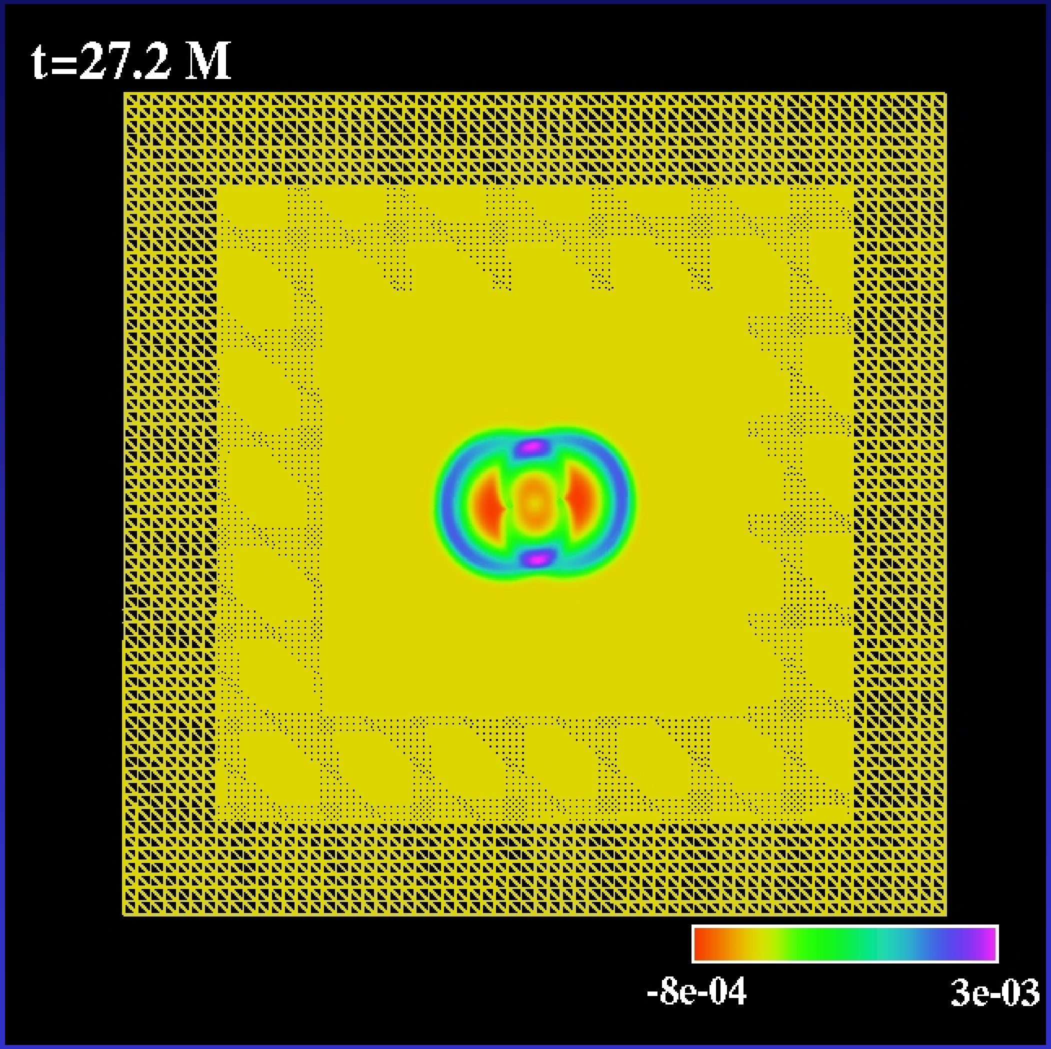


Real component of the Newman-Penrose scalar $\psi_4 . r$, uncompactified coordinates

Summary of computation

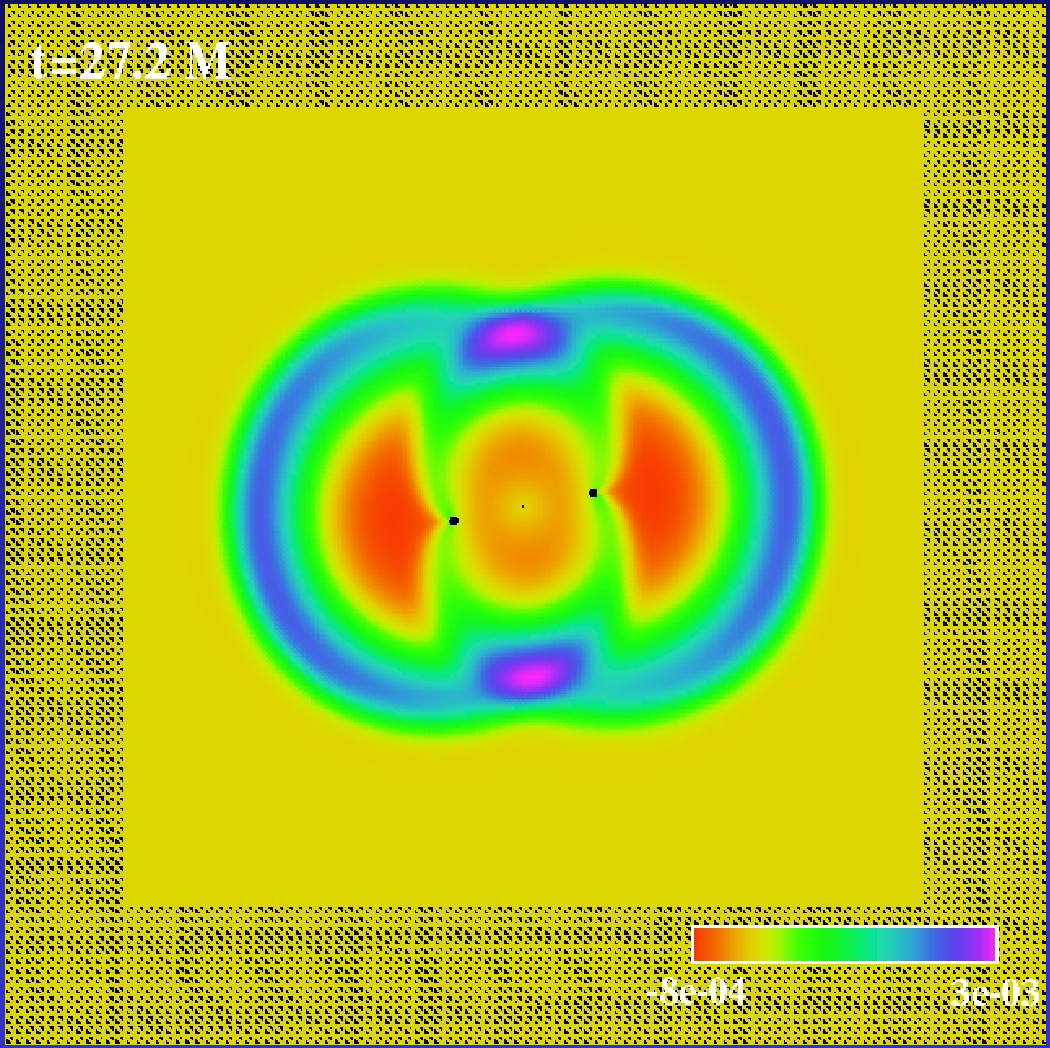
- base grid resolution 48^3
  - 9 levels of 2:1 mesh refinement (*effective* finest grid resolution of 12288^3)
  - so far:
    - ~60,000 time steps on finest level
    - total of around 70,000 CPU hours, first on 48 nodes of UBC's vnp4 cluster, then switched to 128 nodes of Westgrid's Beowulf cluster
  - maximum total memory usage ~ 20GB, disk usage ~ 400GB (and this is very infrequent output!)

Sample mesh structure (different though similar simulation!)



Scalar field ϕ . $r, z=0$ slice

Sample mesh structure (different though similar simulation!)



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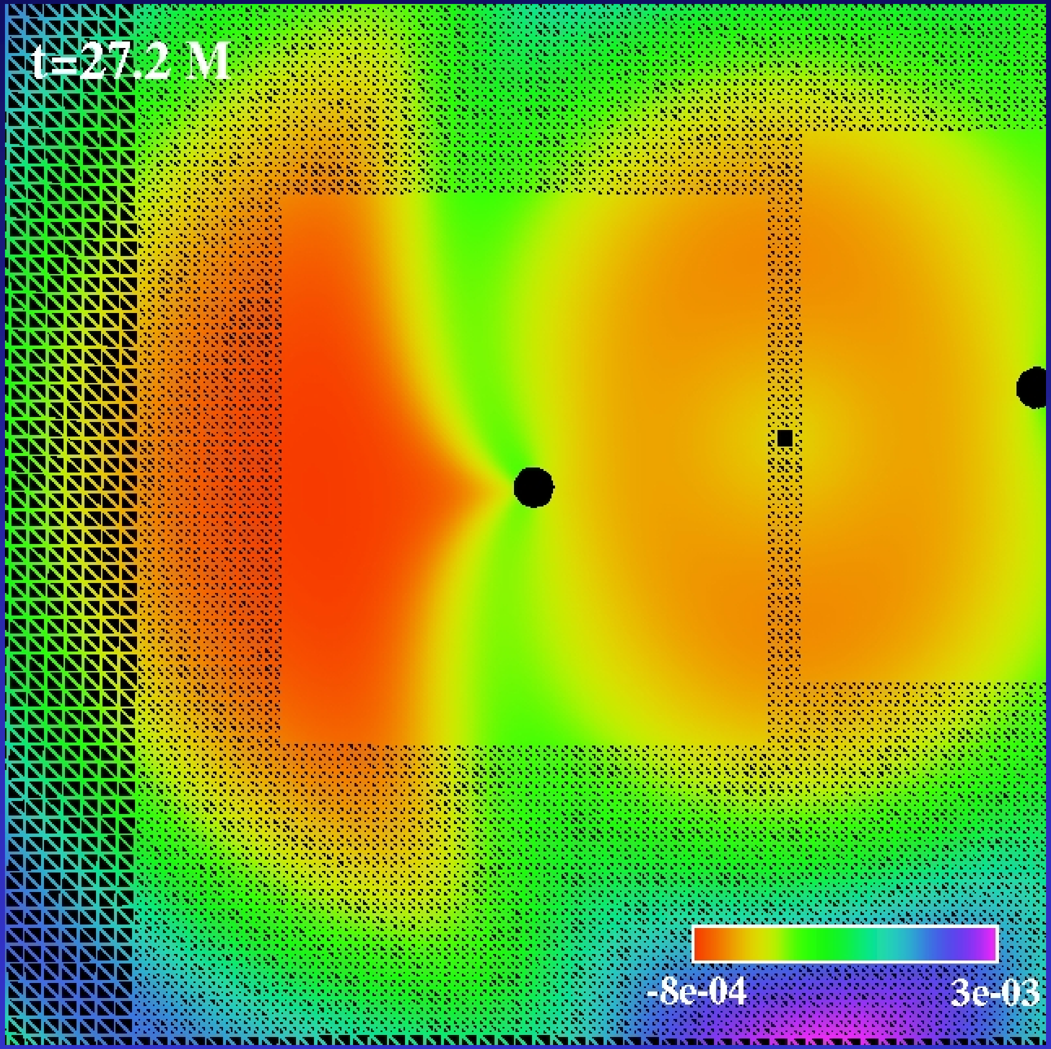
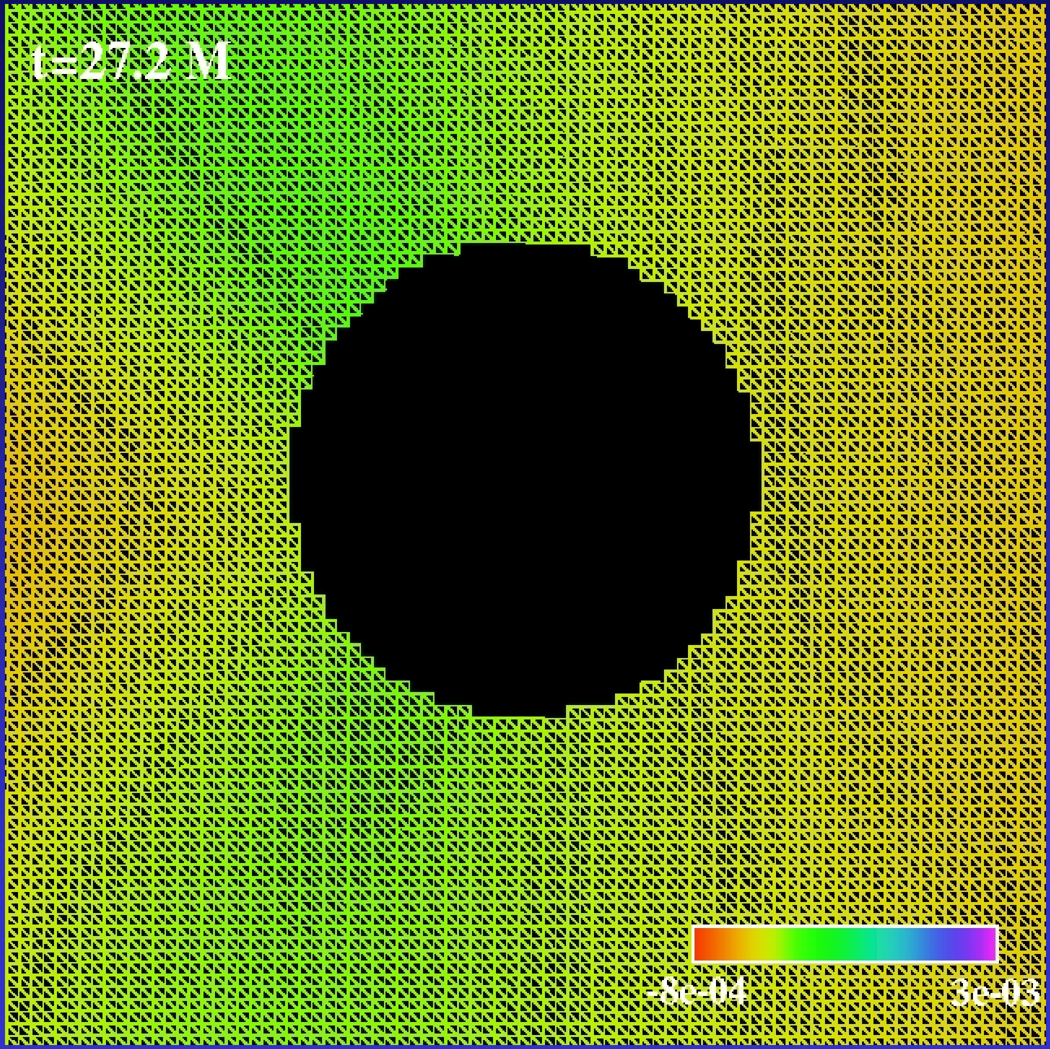


Scalar field $\phi \cdot r$, $z=0$ slice

Sample mesh structure (different though similar simulation!)



Scalar field ϕ . $r, z=0$ slice

Summary

- All indications suggest that this scheme is capable of long term, stable evolutions of binary black hole systems
- Caveats
  - almost prohibitively expensive to run, though working on code optimizations plus finding "good" AMR parameters
  - simple gauge conditions within the harmonic formalism have worked remarkably well for the cases studied so far; though no guarantees that this will continue to be the case for unequal mass ratios, large initial spins, etc…
  - still "tricky" getting the evolution pushed through the merger point
    - indications are this is just a resolution/AH-finder-robustness problem, though because of the "curse of dimensionality" former point is a concern
- What physics can one hope to extract from these simulations in the near future?
  - very broad initial survey of the qualitative features of the last stages of binary mergers
    - pick a handful of orbital parameters (mass ratio, eccentricity, initial separation, individual black hole spins) widely separated in parameters space
    - try to understand the general features of the emitted waves, the total energy radiated, and range of final spins as a function of the initial parameters, plus surprises?