Constraint Preserving Boundary Conditions for BSSN formulation

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Outline

- Introduction to BSSN formulation;
- Constraints of the BSSN system;
- Near-flat linearization;
- Second order in time reduction;
- Evolution of the constraints;
- Constraint preserving boundary conditions;
- Energy estimate for the original BSSN;
- More boundary conditions;

ADM system

Arnowitt–Deser–Misner $3 + 1$ decomposition in vacuum: (lapse a; shift b_i; 3-metric h_{ij}; extrinsic curvature k_{ij}; spatial Ricci tensor R_{ij})

$$\partial_t h_{ij} = -2ak_{ij} + 2D_{(i}b_{j)},$$

$$\partial_t k_{ij} = a[R_{ij} + k^l_l k_{ij} - 2k_{il}k^l_j] + b^l D_l k_{ij} + k_{il}D_j b^l + k_{lj}D_i b^l - D_i D_j a,$$

$$R^i_i + (k^i_i)^2 - k_{ij}k^{ij} = 0, \qquad \text{Hamiltonian const.}$$

$$D^j k_{ij} - D_i k^j_j = 0 \qquad \text{momentum const.}$$

$$R_{ij} = \tfrac{1}{2}h^{pq}(\partial_p\partial_j h_{iq} + \partial_i\partial_q h_{pj} - \partial_p\partial_q h_{ij} - \partial_i\partial_j h_{pq}) + h^{pq}h^{rs}(\Gamma_{ipr}\Gamma_{qjs} - \Gamma_{pqr}\Gamma_{ijs}),$$

$$\Gamma_{ijl} = \frac{1}{2}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}).$$

Ricci Tensor is difficult!

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$$\Gamma_{ijl} = \frac{1}{2}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}).$$

But can be eliminated for the trace of k_{ij}!

BSSN: the Trace of k_{ij}

The evolution equation for $k = k^l_l$ is simple

$$(1) \qquad (\partial_t - b^l D_l)k = a k^{pq} k_{pq} - D^l D_l a$$

(after Hamiltonian constraint was used to eliminate Ricci)

To separate evolution of k from the system, introduce new variables

$$(2) \qquad k = k^i_i, \quad A_{ij} = k_{ij} - \frac{1}{3} h_{ij} k$$

Decompose the system accordingly...

BSSN variables

Motivation: we need a decomposition for h compatible with

$$k = k^i_i, \quad A_{ij} = k_{ij} - \frac{1}{3} h_{ij} k.$$

The new variables

$\varphi = (1/12)\ln(\det h)$ is the conformal factor;

$\tilde{h}_{ij} = \mathrm{e}^{-4\varphi} h_{ij}$, $\tilde{h}^{ij} = \mathrm{e}^{4\varphi} h^{ij}$ are the conformal metric and its inverse;

$k = k_{pq} h^{pq}$, again, the trace of the extrinsic curvature;

$\tilde{A}_{ij} = \mathrm{e}^{-4\varphi} A_{ij}$ the conformal analog of A_{ij}, $\tilde{A}^{ij} = \mathrm{e}^{-4\varphi} A^{ij}$;

$\tilde{\Gamma}_j = \tilde{h}^{pq} \partial_p \tilde{h}_{qj}$, the contracted Christoffel symbol, follows from Ricci tensor decomposition;

Notice:

$$\partial \tilde{h}_{ij} = \mathrm{e}^{-4\varphi} [\partial h_{ij} - \frac{1}{3} h_{ij} h^{pq} \partial h_{pq}]. \quad \text{(trace free)}$$

BSSN formulation

Solving AMD for φ, k, $\tilde{h}$, $\tilde{A}$, $\tilde{\gamma}$ (both Hamiltonian and the momentum constraints are used)

$$(7) \qquad (\partial_t - \tilde{b}^l \partial_l)\varphi = -\tfrac{1}{6}k + \tfrac{1}{6}\partial_s \tilde{b}^s$$

$$(8) \qquad (\partial_t - \tilde{b}^l \partial_l)k = -D^l D_l a + \dots$$

$$(9) \qquad (\partial_t - \tilde{b}^l \partial_l)\tilde{h}_{ij} = -2a\tilde{A}_{ij} + 2\tilde{h}_{s(i}\partial_{j)}\tilde{b}^s - \tfrac{2}{3}\tilde{h}_{ij}\partial_s \tilde{b}^s$$

$$(10) \qquad (\partial_t - \tilde{b}^l \partial_l)\tilde{A}_{ij} = \mathrm{e}^{-4\varphi}[\tfrac{1}{2}a\partial^l \partial_l \tilde{h}_{ij} + a\partial_{(i}\tilde{\Gamma}_{j)} - 2a\partial_i\partial_j\varphi - \partial_i\partial_j a - 2a\tilde{h}_{ij}\partial^l\partial_l\varphi + \tfrac{1}{3}\tilde{h}_{ij}\partial^l\partial_l a] + \dots$$

$$(11) \qquad (\partial_t - \tilde{b}^l \partial_l)\tilde{\Gamma}_i = -\tfrac{4}{3}a\partial_i k + \tfrac{1}{3}\partial_i\partial_s \tilde{b}^s + h_{si}\partial^l\partial_l \tilde{b}^s + \dots$$

Plus

$$(12) \qquad (\partial_t - \tilde{b}^l \partial_l)a = -a^2 k$$

(here $\tilde{b}_j = \mathrm{e}^{-4\varphi}b$; dots stand for terms vanishing in near-flat linearization)

Constraints

The analog of momentum constraint

$$(13) \qquad \tilde{D}^p A_{iq} - \frac{2}{3}\partial_i k = 0;$$

The analog of Hamiltonian constraint

$$(14) \qquad \tilde{R}^i_i - 8\tilde{D}^i\tilde{D}_i\varphi - 8(\partial^i\varphi)(\partial_i\varphi) + \frac{2}{3}k^2 - A^{ij}A_{ij} = 0;$$

An artificial constraint on $\tilde{\Gamma}_j$

$$(15) \qquad \tilde{\Gamma}_j = \tilde{h}^{pq}\partial_p\tilde{h}_{qj}.$$

Linearization around flat space

Space-time metric is a perturbation of the Minkowski metric:
$a = 1 + \alpha$, $b_i = \beta_i$, $h_{ij} = \delta_{ij} + \gamma_{ij}$, $k_{ij} = \kappa_{ij}$, $\alpha, \beta_i, \gamma_{ij}, \kappa_{ij} \approx O(\epsilon)$

Then (up to higher order in perturbations)

$\varphi = \frac{1}{12}\gamma_l^l$;
$k = \kappa := \kappa_l^l$;
$\mathrm{e}^{-4\varphi} = 1 - \frac{1}{3}\gamma_l^l$ and $\mathrm{e}^{4\varphi} = 1 + \frac{1}{3}\gamma_l^l$;
$\tilde{h}_{ij} = \delta_{ij} + \gamma_{ij} - \frac{1}{3}\delta_{ij}\gamma_l^l := \delta_{ij} + \tilde{\gamma}_{ij}$
$\tilde{A}_{ij} = A_{ij} := \kappa_{ij} - \frac{1}{3}\delta_{ij}\kappa$;
$\tilde{\Gamma}_j = \Gamma_j = \partial^l \tilde{\gamma}_{lj}$;

Linearized BSSN

Evolution equations

$$
\begin{aligned}
(E1)\quad & \partial_t\varphi = -\tfrac{1}{6}\kappa + \tfrac{1}{6}\partial^s\beta_s;\\
(E2)\quad & \partial_t\kappa = -\partial^l\partial_l\alpha;\\
(E3)\quad & \partial_t\alpha = -\kappa;\\
(E4)\quad & \partial_t\tilde{\gamma}_{ij} = -2\tilde{A}_{ij} + 2\partial_{(i}\beta_{j)} - \tfrac{2}{3}\delta_{ij}\partial^s\beta_s;\\
(E5)\quad & \partial_t A_{ij} = \tfrac{1}{2}\partial^l\partial_l\tilde{\gamma}_{ij} + \partial_{(i}\Gamma_{j)} - 2\partial_i\partial_j\varphi - 2\delta_{ij}\partial^l\partial_l\varphi - \partial_i\partial_j\alpha + \tfrac{1}{3}\delta_{ij}\partial^l\partial_l\alpha;\\
(E6)\quad & \partial_t\Gamma_i = -\tfrac{4}{3}\partial_i\kappa + \tfrac{1}{3}\partial_i\partial^s\beta_s + \partial^l\partial_l\beta_i;
\end{aligned}
$$

Constraint equations:

(H) $\partial^p\partial^q\tilde{\gamma}_{pq} - 8\partial^l\partial_l\varphi = 0,$ and/or $\partial^l\Gamma_l - 8\partial^l\partial_l\varphi = 0;$

(M) $\partial^j A_{ij} - \frac{2}{3}\partial_i k = 0;$

(art) $\Gamma_j = \partial^l\tilde{\gamma}_{lj}.$

Reduction to second order in time

Differentiating (E5) in time and using the (E1)–(E6) we get

$$(E7) \qquad \partial_t^2 A_{ij} = \partial^l \partial_l A_{ij}.$$

Similarly, differentiating (E2) in time and using (E3) we get

$$(E8) \qquad \partial_t^2 k = \partial^l \partial_l k.$$

Unconstrained evolution problem: Given β, and $\alpha(0)$, $\varphi(0)$, $\kappa(0)$, $\tilde{\gamma}_{ij}(0)$, $A_{ij}(0)$, $\Gamma_i(0)$, calculate $\partial_t A_{ij}(0)$ from (E4) and $\partial_t k(0)$ from (E2), add BCs, solve (E7),(E8) for A_{ij}, κ. Integrate (E1), (E3), (E4), (E6) to calculate α, φ, $\tilde{\gamma}_{ij}$, Γ_i.

Evolution of the constraints

Introduce

$$M_i = \partial^j A_{ij} - \frac{2}{3}\partial_i k;$$
$$H1 = \partial^p \partial^q \tilde{\gamma}_{pq} - 8\partial^l \partial_l \varphi;$$
$$H2 = \partial^l \Gamma_l - 8\partial^l \partial_l \varphi;$$

Notice, that $\partial_t H1 = \partial^j M_j$ so if the momentum constraint is satisfied $(M_j = 0)$ then $H1 \equiv 0$ provided it is zero initially; $\partial_t H2 = 0$ as follows from (E6), (E1).

So, we need to check $M_j = 0$.

Evolution of M_i

The propagation of the momentum constraint is given by

$$(E9) \qquad \partial_t^2 M_j = \partial^l \partial_l M_j$$

the value of $\partial_t M_j(0)$ can be calculated from (E5) and (E2) as

$$\partial_t M_i(0) = -\frac{1}{2}\partial^l \partial_l \partial^m \tilde{\gamma}_{im}(0) + \frac{1}{2}\partial_i \partial^l \Gamma_l(0) + \frac{1}{2}\partial^l \partial_l \Gamma_i(0) - 4\partial_i \partial^l \partial_l \varphi(0).$$

If we can find a set of well-posed boundary conditions for the linearized BSSN system that imply

$$(\frac{\partial}{\partial n} M_j)\partial_t M^j = 0$$

then we are done!

Why $(\frac{\partial}{\partial n}M_j)\partial_t M^j = 0$**?**

Energy argument: contract both sides of (E9) with $\partial_t M^i$, integrate over the domain Ω:

$$\int_\Omega (\partial_t^2 M_j)\partial_t M^j = \int_\Omega (\partial^l \partial_l M_j)\partial_t M^j$$

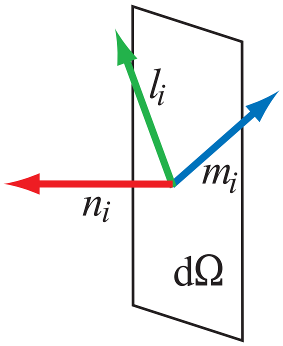
integrate by parts

$$\frac{1}{2}\partial_t[\|\partial_t M_i\|^2 + \|\partial_j M_i\|^2] = \int_{\partial\Omega} (\frac{\partial}{\partial n}M_j)\partial_t M^j.$$

(for example, either $M_i = 0$, or $\frac{\partial}{\partial n}M_i = 0$, or mixed)

$(\frac{\partial}{\partial n} M_j)\partial_t M^j = 0$ via the main variables: the technology

Introduce, orthonormal basis: n_i, m_i, l_i,



Rewrite, $\quad \partial^j u_j = \frac{\partial}{\partial n} u_i n^i + \frac{\partial}{\partial m} u_i m^i + \frac{\partial}{\partial l} u_i l^i,$

$$\partial_i \kappa = \frac{\partial}{\partial n} \kappa n_i + \frac{\partial}{\partial m} \kappa m_i + \frac{\partial}{\partial l} \kappa l_i$$

Decompose

$$A_{ij} = A1\,(n_{(i} m_{j)}) + A2\,(n_{(i} l_{j)}) + A3\,(l_{(i} m_{j)}) + A4\,(l_i l_j - m_i m_j) + A5\,(2 n_i n_j - l_i l_j - m_i m_j).$$

Substitute into $M_i = \partial^j A_{ij} - \frac{2}{3}\partial_i k$ and the result into $(\frac{\partial}{\partial n} M_j)\partial_t M^j = 0$

Simplify ...

$$
\begin{aligned}
&(\frac{\partial}{\partial n}M_i)(\partial_t M^i) \\
&= \frac{\partial}{\partial n}[\frac{1}{2}\frac{\partial}{\partial m}A1 + \frac{1}{2}\frac{\partial}{\partial l}A2 + 2\frac{\partial}{\partial n}A5 - \frac{2}{3}\frac{\partial}{\partial n}\kappa] \\
&\quad \times \partial_t[\frac{1}{2}\frac{\partial}{\partial m}A1 + \frac{1}{2}\frac{\partial}{\partial l}A2 + 2\frac{\partial}{\partial n}A5 - \frac{2}{3}\frac{\partial}{\partial n}\kappa] \\
&+ \frac{\partial}{\partial n}[\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa] \\
&\quad \times \partial_t[\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa] \\
&+ \frac{\partial}{\partial n}[\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa] \\
&\quad \times \partial_t[\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa](= 0\ ???).
\end{aligned}
$$

By direct observation

Either of the two sets of boundary conditions imply $((\partial/\partial n)M^i)(\partial_t M_i) = 0$ on $\partial\Omega$:

$$A1 = 0, \quad A2 = 0, \quad \frac{\partial}{\partial n}A3 = 0, \quad \frac{\partial}{\partial n}A4 = 0, \quad \frac{\partial}{\partial n}A5 = 0, \quad \frac{\partial}{\partial n}\kappa = 0,$$

$$\frac{\partial}{\partial n}A1 = 0, \quad \frac{\partial}{\partial n}A2 = 0, \quad A3 = 0, \quad A4 = 0, \quad A5 = 0, \quad \kappa = 0.$$

(the first set eliminates the second multiplier in the first term of $((\partial/\partial n)M^i)(\partial_t M_i) = 0$ and the first multipliers in the second and third terms (by commuting partial derivatives and using (E9). Second set is verified in a similar way.)

Too restrictive! Needs compatibility with the initial data!

Differential boundary conditions

Require $M_i = 0$:

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial m}A1 + \frac{1}{2}\frac{\partial}{\partial l}A2 + 2\frac{\partial}{\partial n}A5 - \frac{2}{3}\frac{\partial}{\partial n}\kappa &= 0 \\
\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa &= 0 \\
\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa &= 0
\end{aligned}
$$

For example, prescribe $A3$, $A4$, κ, and use $M_i = 0$ as the boundary conditions for the rest.

Evolving boundary conditions

Require $\frac{\partial}{\partial n} M_i n^i = 0$, $M_i l^i = 0$, $M_i m^i = 0$.

Solve for

$$2\frac{\partial^2}{\partial t^2}A5-(\frac{\partial^2}{\partial l^2}+\frac{\partial^2}{\partial m^2})A5-\frac{2}{3}\frac{\partial^2}{\partial t^2}\kappa+\frac{4}{3}(\frac{\partial^2}{\partial l^2}+\frac{\partial^2}{\partial m^2})\kappa = (\frac{\partial^2}{\partial l^2}-\frac{\partial^2}{\partial m^2})A4+\frac{\partial}{\partial l}\frac{\partial}{\partial m}A3,$$

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial n}A1 + \frac{1}{2}\frac{\partial}{\partial l}A3 - \frac{\partial}{\partial m}A4 - \frac{\partial}{\partial m}A5 - \frac{2}{3}\frac{\partial}{\partial m}\kappa &= 0 \\
\frac{1}{2}\frac{\partial}{\partial n}A2 + \frac{1}{2}\frac{\partial}{\partial m}A3 + \frac{\partial}{\partial l}A4 - \frac{\partial}{\partial l}A5 - \frac{2}{3}\frac{\partial}{\partial l}\kappa &= 0
\end{aligned}$$

For example, prescribe $A3$, $A4$, κ, evolve first equation for $A5$, use other two as the inhomogeneous Neumann data on $A1$ and $A2$.