Project 3: The Teukolsky Equation on a Kerr Background in Kerr-Schild Coordinates

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3.1 Introduction

The *Teukolsky equation* describes gravitational, electromagnetic, scalar and neutrino field perturbations of a rotating, or Kerr black hole [1]. The derivation of the Teukolsky equation makes use of the *Newman-Penrose* (NP) formalism [2]. It is beyond the scope of this project to describe either the NP formalism or the derivation of the Teukolsky equation in any detail, however we will briefly outline some of the key concepts of both in the following sections.

3.2 The Newman-Penrose formalism

The NP formalism is a form of tetrad calculus, where the geometry, Einstein field equations and any additional matter equations are projected onto a complex, null tetrad. When an appropriate set of projected quantities, namely the *spin coefficients* are used (which are similar to Ricci rotation coefficients in a traditional tetrad decomposition), the resulting field equations take on a relatively "simple" form (see appendix E of [3], for instance). The null tetrad consists of 2 real null vectors, l^μ and n^μ, a complex null vector m^μ, and its complex conjugate $\bar{m}^\mu$. The normalization of the null vectors is chosen as follows: $\ell^\mu n_\mu = 1$ and $m^\mu \bar{m}_\mu = -1$. The particular choice of null tetrad is application specific. A common choice in "spherical-like" spacetimes (such as Kerr) is to let l^μ and n^μ correspond to ingoing and outgoing, radial null vectors respectively. Then two unit spacelike vectors a^μ and b^μ (vectors in the θ and ϕ directions for example), orthogonal to each other and the real null vectors, are used to construct the complex null vectors via $m^\mu = (1\sqrt{2})(a^\mu - ib^\mu)$.

In a 4-dimensional geometry that satisfies the vacuum Einstein equations, all non-trivial curvature of the metric can be described by the Weyl tensor $C_{\alpha\beta\gamma\delta}$, which is the trace free part of the Riemann tensor. The Weyl tensor has 10 independent components that in the NP formalism are given by 5 complex scalars, called the *Newman-Penrose scalars*, defined via the following projections:

$$\begin{aligned}
\Psi_0 &= -C_{\alpha\beta\gamma\delta} n^\alpha m^\beta n^\gamma m^\delta, \\
\Psi_1 &= -C_{\alpha\beta\gamma\delta} n^\alpha \ell^\beta n^\gamma m^\delta, \\
\Psi_2 &= -C_{\alpha\beta\gamma\delta} \bar{m}^\alpha \ell^\beta n^\gamma m^\delta, \\
\Psi_3 &= -C_{\alpha\beta\gamma\delta} \bar{m}^\alpha \ell^\beta n^\gamma \ell^\delta,
\end{aligned}$$

$$\Psi_4 \;=\; -C_{\alpha\beta\gamma\delta}\bar{m}^\alpha \ell^\beta \bar{m}^\gamma \ell^\delta. \tag{1}$$

One of the interesting properties of the NP scalars is how they characterize the so-called "peeling" property [4] of an asymptotically flat spacetime: with no radiation coming in from infinity, the leading order behavior of the NP scalars are

$$\begin{aligned}
\Psi_0 &= O(r^{-5}), \\
\Psi_1 &= O(r^{-4}), \\
\Psi_2 &= O(r^{-3}), \\
\Psi_3 &= O(r^{-2}), \\
\Psi_4 &= O(r^{-1}).
\end{aligned} \tag{2}$$

Here r is the distance from some region of strong-field gravity, and the tetrad used is as described earlier, namely ℓ^μ and n^μ are tangent to radially outgoing and ingoing null vectors respectively. Equations (2) show that in the vicinity of the source *all* of the scalars are relevant in describing the gravitational wave content of the spacetime; as one moves away from the source, one-by-one the scalars become irrelevant (they "peel" off). The only NP scalar that dies off sufficiently slowly to be relevant at infinity, and hence carry gravitational wave information, is Ψ_4. In axisymmetry Ψ_4 is real; a non-zero imaginary part would be associated with angular momentum carried by the waves.

The energy flux of gravitational waves radiated by a source can be calculated at large distances from the source via [5]

$$\frac{dM}{dt} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{r=\mathrm{const}} \left[\int_0^t \Psi_4 dt\right]^2 r^2 d\Omega, \tag{3}$$

where M is the ADM mass [8] of the spacetime, $d\Omega$ denotes integration over the unit sphere, and we assume that the outgoing radiation is zero before $t = 0$.

3.3 The Teukolsky Equation

The Teukolsky equation makes use of the NP formalism to arrive at a sufficient set of linear PDEs that describe small perturbations of a Kerr black hole. In traditional perturbation theory, one would consider perturbations of the metric directly, i.e. let $g_{\mu\nu} = g^A_{\mu\nu} + h^B_{\mu\nu}$, where $g^A_{\mu\nu}$ is the background metric, and $h^B_{\mu\nu}$ the small, perturbed part. One would then find the linear equations governing the perturbations. Taking such an approach in a Kerr background does not lead to a separable set of PDEs. However, using the NP formalism, Teukolsky was able to find simple, separable PDEs that describe perturbations of Kerr black holes by gravitational, electromagnetic, scalar and neutrino waves. Moreover, the gravity wave perturbations are given directly in terms of Ψ_4 and Ψ_0, and so in some sense are gauge invariant descriptions of the perturbations.

To do perturbation theory in the NP formalism, one does not directly perturb the metric; rather one perturbs the null tetrad, derivative operators, and the geometric scalars of the formalism, i.e. $l_\mu = l^A_\mu + l^B_\mu$, $\Psi_4 = \Psi^A_4 + \Psi^B_4$, etc ... Again, the A variables denote background quantities, and B the arbitrary functions of the perturbation. These are then plugged into the NP equations, and only terms that are first order in B are kept. For the Kerr geometry [1], the resulting NP equations for Ψ_4^B and Ψ_0^B (outgoing and ingoing radiation in the asymptotic region, respectively) decouple from the rest. For the matter fields, the equations of motion for the scalar field, for two of the projected components of the Maxwell tensor, and for the projected components of the neutrino spinor field decouple.

3.4 The Teukolsky Equation for Gravitational Wave Perturbations

To study gravitational wave perturbations of rotating black holes, in particular the waves that are radiated to infinity, we want to solve the Teukolsky equation for Ψ_4:

$$[(\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)(D + 4\epsilon - \rho) - (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\Psi_2]\,\Psi_4 = 0, \quad (4)$$

where the quantities in brackets are computed using the background geometry, and Ψ_4 is zero for the unperturbed Kerr metric, so we drop all the superscripts A and B for simplicity. The differential operators in the above equation are $D = l^\mu\partial_\mu$, $\Delta = n^\mu\partial_\mu$, $\delta = m^\mu\partial_\mu$ and $\delta^* = \bar{m}^\mu\partial_\mu$; the remaining symbols denote various Newman-Penrose quantities of the background spacetime that we will shortly define.

As originally introduced by Teukolsky, the background spacetime is given by the Kerr metric in Boyer–Lindquist coordinates

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \left(\frac{4Mar\sin^2\theta}{\Sigma}\right)dtd\varphi - \left(\frac{\Sigma}{\triangle}\right)dr^2 - \Sigma d\theta^2 - \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)d\varphi^2, \quad (5)$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\triangle = r^2 - 2Mr + a^2$ (and should not be confused with the Newman–Penrose quantity $\Delta = n^\mu\partial_\mu$), with M and a the mass and angular momentum per unit mass of the black hole respectively. We consider the Kinnersley tetrad

$$\begin{aligned}
l^\mu &= \left[\left(r^2 + a^2\right)/\triangle, 1, 0, a/\triangle\right], \\
n^\mu &= \left[r^2 + a^2, -\triangle, 0, a\right]/(2\Sigma), \\
m^\mu &= \left[ia\sin\theta, 0, 1, i/\sin\theta\right]/(\sqrt{2}(r + ia\cos\theta)),
\end{aligned}$$

where we write the components of a vector T^μ as $[T^t, T^r, T^\theta, T^\phi]$. Then the background Newman–Penrose quantities evaluate to

$$\begin{aligned}
\rho &= -1/(r - ia\cos\theta), \\
\beta &= -\rho^*\cot\theta/(2\sqrt{2}), \\
\pi &= ia\rho^2\sin\theta/\sqrt{2}, \\
\tau &= -ia\rho\rho^*\sin\theta/\sqrt{2}, \\
\mu &= \rho^2\rho^*\triangle/2, \\
\gamma &= \mu + \rho\rho^*(r - M)/2, \\
\alpha &= \pi - \beta^*, \\
\Psi_2 &= M\rho^3.
\end{aligned}$$

[1] More generally any Petrov type D vacuum background when the unperturbed tetrad is chosen to lie along the repeated principle null directions of the Weyl tensor

With this choice of coordinates and null tetrad, equation (4) becomes

$$\left[\frac{(r^2+a^2)^2}{\triangle} - a^2\sin^2\theta\right]\frac{\partial^2}{\partial t^2}\Psi + \frac{4Mar}{\triangle}\frac{\partial^2}{\partial t\partial\varphi}\Psi + \left[\frac{a^2}{\triangle} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2}{\partial\varphi^2}\Psi$$
$$-\triangle^2\frac{\partial}{\partial r}\left(\frac{1}{\triangle}\frac{\partial}{\partial r}\Psi\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\Psi\right) + 4\left[\frac{M(r^2-a^2)}{\triangle} - r - ia\cos\theta\right]\frac{\partial}{\partial t}\Psi$$
$$+4\left[\frac{a(r-M)}{\triangle} + \frac{i\cos\theta}{\sin\theta}\right]\frac{\partial}{\partial\varphi}\Psi + (4\cot^2\theta + 2)\Psi = 0, \tag{6}$$

where $\Psi = (r - ia\cos\theta)^4\Psi_4$.

3.5 The Teukolsky Equation in ingoing–Kerr–Schild coordinates

The ingoing-Kerr-Schild coordinates $(\tilde{V}, r, \theta, \varphi)$ are defined through a redefinition of the time coordinate of the Boyer–Lindquist coordinates via

$$\tilde{V} = t + r^* \tag{7}$$
$$\tilde{\varphi} = \varphi + \int \frac{a}{\triangle} dr \tag{8}$$

where r^* is the natural generalization to the Kerr case of the usual Schwarzschild "tortoise" coordinate, and is defined by

$$r^* = \int \frac{r^2+a^2}{r^2 - 2Mr + a^2} dr. \tag{9}$$

It is also useful to introduce the coordinate $\tilde{t} = \tilde{V} - r$. The Teukolsky equation will then be rewritten in the $(\tilde{t}, r, \theta, \tilde{\varphi})$ coordinates. The Kerr metric in these coordinates reads

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right)d\tilde{t}^2 - \left(1 + \frac{2Mr}{\Sigma}\right)dr^2 - \Sigma d\theta^2 - \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)d\tilde{\varphi}^2$$
$$-\left(\frac{4Mr}{\Sigma}\right)d\tilde{t}dr + \left(\frac{4Mra\sin^2\theta}{\Sigma}\right)d\tilde{t}d\tilde{\varphi} + 2a\sin^2\theta\left(1 + \frac{2Mr}{\Sigma}\right)drd\tilde{\varphi}. \tag{10}$$

In addition to changing coordinates, we also need to change tetrads. The usual Kinnersley tetrad is singular at the horizon, and therefore leads to a Teukolsky equation that is singular. However, as pointed out in [6], this problem can be avoided by the use of the following "rescaled" tetrad vectors:

$$\begin{aligned} l^\mu &= [\triangle + 4Mr, \triangle, 0, 2a], \\ n^\mu &= [\frac{1}{2\Sigma}, -\frac{1}{2\Sigma}, 0, 0]. \end{aligned}$$

The orthogonality properties of the tetrad remain unchanged by this rescaling. After extremely boring but straightforward manipulations, the new form of the Teukolsky equation is given by

$$
\begin{aligned}
&(\Sigma + 2Mr)\frac{\partial^2}{\partial \tilde{t}^2}\Psi - \triangle\frac{\partial^2}{\partial r^2}\Psi - 6(r-M)\frac{\partial}{\partial r}\Psi \\
&-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\Psi\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\tilde{\varphi}^2}\Psi - 4Mr\frac{\partial^2}{\partial\tilde{t}\partial r}\Psi - 2a\frac{\partial^2}{\partial r\partial\tilde{\varphi}}\Psi \\
&+\left(\frac{4i\cot\theta}{\sin\theta}\right)\frac{\partial}{\partial\tilde{\varphi}}\Psi \\
&-(4r + 4ia\cos\theta + 6M)\frac{\partial}{\partial\tilde{t}}\Psi + 2(3\cot^2\theta - \csc^2\theta)\Psi = 0.
\end{aligned} \tag{11}
$$

where $\Psi = (r - ia\cos(\theta))^4\Psi_4$. To reduce the equation to a 2+1 form, we introduce the following ansatz:

$$
\Psi(t, r, \theta, \varphi) = \sum_m \Psi_m(t, r, \theta)e^{im\tilde{\varphi}}\,. \tag{12}
$$

The resulting Teukolsky equation for each integer mode m is then given by

$$
\begin{aligned}
&(\Sigma + 2Mr)\frac{\partial^2}{\partial\tilde{t}^2}\Psi_m - \triangle\frac{\partial^2}{\partial r^2}\Psi_m - (2aim + 6r - 6M)\frac{\partial}{\partial r}\Psi_m \\
&-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\Psi_m\right) - 4Mr\frac{\partial^2}{\partial\tilde{t}\partial r}\Psi_m - (4r + 4ia\cos\theta + 6M)\frac{\partial}{\partial\tilde{t}}\Psi_m \\
&+(4\cot^2\theta - 2 + m^2\csc^2\theta - 4m\cot\theta\csc\theta)\Psi_m = 0.
\end{aligned} \tag{13}
$$

Hereafter, we will drop the tildes on the Kerr-Schild coordinates; i.e. (t, r, θ, φ) subsequently refer to Kerr-Schild values.

3.7 The Azimuthally-Separated Wave Equation in Spherical-Polar Coordinates

In this section we describe a simple wave equation, as well its solution via finite difference techniques and RNPL. The discretization method, as well as the RNPL code, can serve as a template for your solution of the Teukolsky equation.

We work in Minkowski spacetime, and adopt the usual spherical polar coordinates, (t, r, θ, φ); then the line element is

$$
ds^2 = -dt^2 + dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\varphi^2 \tag{14}
$$

We adopt the following ansatz for the real scalar (wave), field, $\phi(t, r, \theta, \varphi)$

$$
\phi(t, r, \theta, \varphi) = e^{im\varphi}\,\Phi(t, r, \theta) \tag{15}
$$

where $i = \sqrt{-1}$ and $m = 0, 1, \cdots$ is the azimuthal "quantum" number of the particular scalar field configuration under consideration.

Problem 3a) Using the following formula for the D'Alembertian operator:

$$\nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \phi}{\partial x^\mu} \right) \tag{16}$$

where g is the determinant of the 4-metric, show that with the ansatz (15), the wave equation

$$\nabla^\mu \nabla_\mu \Phi = 0 \tag{17}$$

may be written as

$$-\frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Phi}{\partial \theta} \right) - \frac{m^2}{r^2 \sin^2\theta} \Phi = 0\,. \tag{18}$$

We cast (18) in first-order-in-time form by introducing the auxiliary variable, $\Pi(t, r, \theta)$, defined by

$$\Pi \equiv \frac{\partial \Phi}{\partial t} \tag{19}$$

With this definition, (18) becomes the following system

$$\frac{\partial \Phi}{\partial t} = \Pi \tag{20}$$

$$\frac{\partial \Pi}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Phi}{\partial \theta} \right) - \frac{m^2}{r^2 \sin^2\theta} \Phi \tag{21}$$

We solve (20-21) on the domain

$$0 \le t \le t_{\rm final} \qquad r_{\rm min} \le r \le r_{\rm max} \qquad 0 \le \theta \le \pi \tag{22}$$

where $t_{\rm final}, r_{\rm min}$ and $r_{\rm max}$ are specified positive values. Note that we purposefully exclude $r = 0$ from the domain in order to avoid having to deal with regularity issues at the origin, and in preparation for the solution of the Teukolsky equation, where an identical coordinate domain will be adopted. Equations (20-21) must be augmented with appropriate initial and boundary conditions. We impose homogeneous Dirichlet conditions at $r = r_{\rm min}$, and an approximate outgoing radiation condition, based on a $1/r$ falloff in Φ, at $r = r_{\rm max}$. At $\theta = 0$ and $\theta = \pi$ we use the fact that $\Phi \sim (\sin\theta)^m$. Thus, we have

$$\Phi(t, r_{\rm min}, \theta) = 0 \tag{23}$$

$$\Pi(t, r_{\rm min}, \theta) = 0 \tag{24}$$

$$\left[\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial r} + \frac{\Phi}{r} \right]_{r=r_{\rm max}} = 0 \tag{25}$$

$$\left[\frac{\partial \Pi}{\partial t} + \frac{\partial \Pi}{\partial r} + \frac{\Pi}{r} \right]_{r=r_{\rm max}} = 0 \tag{26}$$

$$\Phi(t, r, 0) = \Phi(t, r, \pi) = 0 \qquad m > 0 \tag{27}$$

$$\Pi(t, r, 0) = \Pi(t, r, \pi) = 0 \qquad m > 0 \tag{28}$$

$$\left[\frac{\partial \Phi}{\partial \theta} \right]_{\theta=0,\pi} = 0 \qquad m = 0 \tag{29}$$

$$\left[\frac{\partial \Pi}{\partial \theta} \right]_{\theta=0,\pi} = 0 \qquad m = 0 \tag{30}$$

We are essentially free to pick arbitrary initial data for $\Phi(0,r,\theta)$ and $\Pi(0,r,\theta)$; here we choose a time-symmetric Gaussian (in both r and θ) pulse, which is "modulated" by an appropriate power of $\sin(\theta)$ to ensure that the angular regularity conditions are satisfied initially:

$$\begin{aligned}
\Phi(0,r,\theta) &= A\exp(-((r-r_0)/\delta r)^2)\exp(-((\theta-\theta_0)/\delta\theta)^2)\sin^m\theta, && (31)\\
\Pi(0,r,\theta) &= 0, && (32)
\end{aligned}$$

where $A, r_0, \delta_r, \theta_0$ and $\delta\theta$ are adjustable parameters.

Before proceeding to a description of our specific discretization of the set of PDEs (20-21) and initial/boundary conditions (23-32), we make a brief digression to discuss aspects of a formal calculus of difference operators for the construction of finite difference systems. The use of such a formalism often provides a compact and elegant way of deriving and expressing finite difference formula, and is directly supported by RNPL. We illustrate the general ideas using one-dimensional operators (spatial coordinate, x), but the generalization to two or more dimensions is straightforward.

At an abstract level, difference operators are maps that take grid functions into grid functions. Let f_j denote an arbitrary grid function defined on some grid, x_j, that has uniform grid spacing Δx. Then the identity operator I, the forward translation operator T^+ and the backward translation operator T^-, are defined via their action on f_j as follows:

$$\begin{aligned}
I f_j &= f_j && (33)\\
T^+ f_j &= f_{j+1} && (34)\\
T^- f_j &= f_{j-1} && (35)
\end{aligned}$$

The above three operators constitute the basic building blocks from which most of the operators that we will use in the discretization of the wave equation will be constructed. Using them, we can now define the first forward and backward divided differences, D^+ and D^-, respectively:

$$\begin{aligned}
D^+ f_j &= \frac{f_{j+1}-f_j}{\Delta x} && (36)\\
D^- f_j &= \frac{f_j-f_{j-1}}{\Delta x} && (37)
\end{aligned}$$

Also useful are the forward and backward averaging operators, μ^+ and μ^-:

$$\begin{aligned}
\mu^+ f_j &= \frac{1}{2}(f_j+f_{j+1}) && (38)\\
\mu^- f_j &= \frac{1}{2}(f_j+f_{j-1}) && (39)
\end{aligned}$$

Using these first divided difference and averaging operators, we can now readily generate second-order ($O(\Delta x^2)$) centred approximations for differential expressions such as

$$\frac{d}{dx}f(x), \qquad \frac{d^2}{dx^2}f(x), \qquad \frac{d}{dx}\left(g(x)\frac{d}{dx}f(x)\right). \qquad (40)$$

Problem 3b) Verify the following:

$$\mu^+ D^- f_j \equiv D^0 f_j = \frac{f_{j+1} - f_{j-1}}{2\Delta x} = \left[\frac{d}{dx} f(x)\right]_{x=x_j} + O(\Delta x^2) \tag{41}$$

$$D^+ D^- f_j = \frac{f_{j+1} - 2f_j + f_{j-1}}{\Delta x^2} = \left[\frac{d^2}{dx^2} f(x)\right]_{x=x_j} + O(\Delta x^2) \tag{42}$$

$$\begin{aligned} D^+ \left(\left(\mu^- g_j\right) \left(D^- f_j\right) \right) &= \frac{(g_{j+1} + g_j)(f_{j+1} - f_j) - (g_j + g_{j-1})(f_j - f_{j-1})}{2\Delta x^2} \\ &= \left[\frac{d}{dx}\left(g(x)\frac{d}{dx} f(x)\right)\right]_{x=x_j} + O(\Delta x^2) \end{aligned} \tag{43}$$

As discussed in Section 1.10 of [7], when solving wave-type equations using finite difference approximations (FDAs) that are naturally non-dissipative (and FDAs that use second-order centred differencing and space and Crank Nicholson time stepping, fall into this class), it is often advantageous to add explicit dissipation to the scheme in order to damp high frequency solution components. Damping of such components is motivated by the fact that high frequency modes are often responsible for stability problems, and justified by the fact that the discrete phase speeds of the shortest wavelength modes have little, if any, relation to the corresponding continuum values.

For the case of $O(h^2)$ approximations of wave equations, so-called Kreiss-Oliger (KO) dissipation often proves effective, and can be illustrated in the context of the simple advection equation:

$$\frac{\partial}{\partial t} u(t,x) = a \frac{\partial}{\partial x} u \tag{44}$$

that can be differenced in Crank-Nicholson fashion as follows:

$$D_t^+ u_j^n = a\mu_t^+ \left(D_x^0 u_j^n\right) \tag{45}$$

Here, the subscript/superscript on the grid function is the spatial/temporal index respectively, while the subscripts on the difference and averaging operators indicate in which of the two coordinate directions the operators act. We add KO dissipation to the above FDA as follows:

$$D_t^+ u_j^n = a\mu_t^+ \left(D_x^0 u_j^n\right) - \frac{\epsilon \Delta x^4}{16\Delta t} D_x^+ D_x^+ D_x^- D_x^- u_j^n \,. \tag{46}$$

In the above expression, ϵ is an adjustable parameter that controls the amount of dissipation, while the factor of 16 is a "normalization" constant, chosen so that the normal range of ϵ is $0 \le \epsilon < 1$. (the upper bound of ϵ is generally limited by stability considerations—too much of a good thing can ruin the scheme!)

Problem 3c) Verify the following:

$$D_x^+ D_x^+ D_x^- D_x^- u_j^n = \frac{u_{j+2}^n - 4u_{j+1}^n + 6u_j^n - 4u_{j-1}^n + u_{j-2}^n}{\Delta x^4} = \left[\frac{\partial^4}{\partial x^4} u\right]_{x=x_j} + O(\Delta x^2) \tag{47}$$

and argue that (46) remains an $O(\Delta x^2, \Delta t^2) = O(h^2)$ approximation to (44); i.e. addition of KO dissipation does not change the leading order truncation error of the scheme.

Note that it is clear from the above result that on a mesh x_j, $j = 1, 2, \cdots, N_x$, the KO dissipation operator can only be applied at $j = 3, 4, \cdots, N_x - 2$. The fact that dissipation cannot be applied for $j = 2$ or $j = N_x - 1$ frequently does not pose a problem in practice, although this is certainly not a universal rule. We also observe that for multi-space-dimensional schemes, we can often simply apply KO dissipation in each of the spatial coordinate directions, and typically using the same value of ϵ. Finally, for subsequent notational convenience we define the operator Z_x^ϵ via

$$Z_x^\epsilon u_j^n = -\frac{\epsilon \Delta x^4}{16 \Delta t} D_x^+ D_x^+ D_x^- D_x^- u_j^n \tag{48}$$

so that (46) becomes

$$D_t^+ u_j^n = a \mu_t^+ \left(D_x^0 u_j^n \right) + Z_x^\epsilon u_j^n \tag{49}$$

We now end our digression, and return to the differencing of (20-21) and (23-32). We discretize the continuum domain (22) by introducing a uniform mesh $(t^n, r_i, \theta_j), n = 0, 1, \ldots,\ i = 1, 2, \ldots, N_x,\ j = 1, 2, \ldots, N_\theta$. We then discretize the first order form of the wave equation (20-21) as follows:

$$\begin{aligned}
D_t^+ \Phi_{i,j}^n &= \mu_t^+ \Pi_{i,j}^n + (Z_r^\epsilon + Z_\theta^\epsilon) \Phi_{i,j}^n & (50)\\
D_t^+ \Pi_{i,j}^n &= \mu_t^+ \left[\frac{1}{r_i^2} D_r^+ \left(\left(\mu_r^- r_i^2 \right) \left(D_r^- \Phi_{i,j}^n \right) \right) + \frac{1}{r_i^2 \sin\theta_j} D_\theta^+ \left(\left(\mu_\theta^- \sin\theta_j \right) \left(D_\theta^- \Phi_{i,j}^n \right) \right) \right] - & (51)\\
&\quad \frac{m^2}{r_i^2 \sin^2\theta_j} \mu_t^+ \Phi_{i,j}^n + (Z_r^\epsilon + Z_\theta^\epsilon) \Pi_{i,j}^n
\end{aligned}$$

Again, we note that due to the width of the dissipation operators, (50-51) can only be applied for $i = 3, 4, \cdots, N_r - 2$, $j = 3, 4, \cdots N_\theta - 2$. Along the penultimate "rows" and "columns" of the two dimensional lattice (i.e. at $i = 2$, $i = N_r - 1$, $j = 2$, $j = N_\theta - 1$) we apply dissipation *only* in the dimension where it makes sense (e.g. along the θ direction at $i = 2$). The full details should be clear from examining the RNPL code, `wave2d_sph_rnpl`, which will be discussed shortly.

We also need to describe the discretization of the initial and boundary conditions. The initial conditions (31-32) are trivial to discretize, as are the Dirichlet conditions (23-24) and (27-28). We use a Crank-Nicholson discretization of the outgoing radiation conditions (25-26), approximating the spatial derivatives with an $O(\Delta r^2)$ backwards FDA. A precisely analogous procedure is used for the treatment of the angular Neumann conditions (29-30).

As discussed in Section 1.9 of [7], in order to check the correctness of a particular FDA and its implementation, it is often useful to use a technique based on *independent residual evaluation*. Briefly, this involves coding a second, *distinct* (i.e. independent) FDA of the original system of PDEs, applying it to the numerical solution to the original FDA, and demonstrating that the residuals thus generated vanish with the expected power of the mesh spacing in the continuum limit. Our implementation of the FDA described above includes such a computation, but only with respect to the PDE (21), (i.e. we do not code residual evaluators for (20), or for the boundary conditions), and leave the temporal discretization intact.

For the purposes of generating the second discretization of (21), it is convenient to "expand" the terms involving spatial derivatives, yielding

$$\frac{\partial \Pi}{\partial t} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r^2 \tan\theta}\frac{\partial \Phi}{\partial \theta} - \frac{m^2}{r^2 \sin^2\theta}\Phi\,. \tag{52}$$

We then replace all spatial derivatives with centred, $O(h^2)$ FDAs, and obtain our independent discretization:

$$D_t^+ \hat{\Pi}_{i,j}^n = \mu_t^+ \left[\left(D_r^+ D_r^- + \frac{2}{r_i} D_r^0 + \frac{1}{r_i^2} D_\theta^+ D_\theta^- + \frac{1}{r_i^2 \tan\theta_j} D_\theta^+ - \frac{m^2}{r_i^2 \sin^2\theta_j} \right) \hat{\Phi}_{i,j}^n \right] \tag{53}$$

Note that we have put carets (hats) on the grid functions in the above to emphasize that the solutions of the two discretizations (51) and (53) are distinct.

Problem 3d) Download the `wave2d_sph` distribution from the lab web page, compile it, then note the following features about the RNPL code.

1. Definition of a multi-dimensional coordinate system:

```
polar coordinates t,r,th

uniform polar grid g1 [1:Nr][1:Nth] {rmin:rmax}{thmin:thmax}
```

2. Definition of multi-dimensional difference operators. For example, the following two statements define the first forward divided difference operators, D_r^+ and D_θ^+, discussed above:

```
operator DP(f,r)       := (<0>f[1][0] - <0>f[0][0])/(dr)

operator DP(f,th)      := (<0>f[0][1] - <0>f[0][0])/(dth)
```

Note that both of the above operators are ostensibly named `DP`. However, the "true" name of the operator `DP(f,r)` is essentially everything *but* the formal argument `f`, i.e. `DP( ,r)`. Thus the independent variable (or variables, for higher order difference operators), in effect, constitutes part of the name of a difference operator, allowing the same prefix (`DP`) in this case to be used in an arbitrary number of operator definitions.

Similarly the first backward divided differences, D_r^-, D_θ^-, and the backward averaging operators, μ_r^-, μ_θ^-, are defined by

```
operator DM(f,r)       := (<0>f[0][0] - <0>f[-1][0])/(dr)
operator DM(f,th)      := (<0>f[0][0] - <0>f[0][-1])/(dth)

operator MUM(f,r)      := (<0>f[0][0] + <0>f[-1][0])/(2)
operator MUM(f,th)     := (<0>f[0][0] + <0>f[0][-1])/(2)
```

3. Composition (nesting) of difference operators. The following example implements operators of the form (43) with $g(r) = r^2$ and $g(\theta) = \sin(\theta)$, (and trivially modified by an overall division by another function of the coordinates in both cases):

```
operator DOC(f,r,r)   := DP(MUM(r^2,r)*DM(f,r),r)/r^2

operator DOC(f,th,th) := DP(MUM(sin(th),th)*DM(f,th),th)/r^2/sin(th)
```

4. Definition of indexing regions in residual statements. Example:

```
[3:Nr-2][2:2]         := DCN(ph,t) = MUD(DISS_R(ph,t),t) + MU(pi,t);
[3:Nr-2][3:Nth-2]     := DCN(ph,t) = MUD(DISS(ph,t),t)   + MU(pi,t);
[3:Nr-2][Nth-1:Nth-1] := DCN(ph,t) = MUD(DISS_R(ph,t),t) + MU(pi,t);
```

   Note that the parts of the statement preceding the `:=` sign in the above each define a rectangular sub-region of the total index space `[1:Nr][1:Nth]` at which the corresponding difference expression is applied. In the above instance, the single difference between the second, and the first and third, statements, is that in the latter case, we apply dissipation *only* in the r-direction. It is absolutely typical in multi-dimensional RNPL code to have the majority of residual definitions dealing with boundary, or near-boundary cases.

5. Logical (boolean) evaluations in residual statements. The following statements define residuals for `ph` that are a) Dirichlet when $m > 0$, and b) Neumann otherwise (i.e. for $m = 0$):

```
[1:Nr][1:1]     := if (m>0) then <1>ph[0][0]=0 else AD(DF(ph,th),t)=0;
[1:Nr][Nth:Nth] := if (m>0) then <1>ph[0][0]=0 else AD(DB(ph,th),t)=0;
```

   Note that RNPL uses "C-style" logical comparisons, in particular a test for equality is accomplished with the `==` binary operator.

Of course, if any of the above features (or anything else about the RNPL code is confusing and/or unclear to you), you can ask one of the lab instructors for assistance.

Problem 3e) The `wave2d_sph` distribution includes sub-directories `run_m0`, `run_m1` and `run_m2` that contain parameter files for sample evolutions with $m = 0$, $m = 1$ and $m = 2$, respectively. Using the local `vn.physics.ubc.ca` Linux cluster (see the lab page for usage instructions), run the code in these directories, investigating convergence and noting the principal similarities and differences in the solution features for the different values of m.

Note: In order to maintain reasonable turn-around times in performing the convergence tests, we suggest that you modify the parameter files so that a maximum of 128 iterations are taken, and so that output is performed at every time step. Also note that you can get a gross summary of the convergence of specific grid functions using the `cvtestsdf` command. Typical usage and output is as follows:

```
vnfe1% cvtestsdf m0_ph_0.sdf m0_ph_1.sdf m0_ph_2.sdf
0 0
0.106971 3.98327
0.213942 5.26145
0.320913 6.05094
0.427883 6.67171
0.534854 7.33581
0.641825 8.20072
0.748796 9.26336
0.855767 9.34238
        .
        .
        .
```

`cvtestsdf` outputs two columns: the first is the integration time, the second is the convergence factor, computed, in this case, by dividing the norm of the difference between the level-0 and level-1 solutions by the norm of the difference between the level-1 and level-2 solutions. In the continuum limit, and assuming that the code is second-order convergent, the convergence factor should tend to a value of 4 for all times. We observe that, particularly for the case of initial data that is specified in closed-form ("analytic"), and on coarse grids, it is not unusual to see large fluctuations in the convergence factor at very early times. As usual, the more discretization levels that can be employed in a convergence test, the better, but also note that for 2-D calculations, each finer level is a factor of 8 more costly to compute, so, especially in the lab setting, there is a practical limit to how stringently we can convergence test such codes.

IMPORTANT!! Because global disk space on the `vn` cluster is limited, and because there are a large number of student groups, we may eventually have to ask you to run your multi-dimensional calculations in the `/tmp/$USER` (where `$USER` is your account name) directories on the nodes.

PROBLEM 3f) Write an RNPL program to solve the Teukolsky equation on a Kerr background in Kerr-Schild coordinates (13).

We suggest that you attack this problem using the following steps:

1. Using `wave2d_sph` as a template, start by writing an RNPL code that implements a straightforward discretization of (13). In doing this, you should also do the following:
    - Re-express (13) as a *pair of* real-valued equations by taking its real and imaginary parts.
    - Re-write the resulting pair of equations as a system of *four* first-order-in-time equations, by introducing auxiliary variables (say, $\Pi_{\rm Re}$ and $\Pi_{\rm Im}$) for the time derivatives of the real and imaginary parts (say, $\Phi_{\rm Re}$ and $\Phi_{\rm Im}$) of the wave field.

- Use the following regularity conditions at $\theta = 0$ and $\theta = \pi$:

$$\left[\frac{\partial}{\partial\theta}F(t,r,\theta)\right]_{\theta=0,\pi} = 0 \qquad \text{for } m \text{ even} \tag{54}$$

$$\left[F(t,r,\theta)\right]_{\theta=0,\pi} = 0 \qquad \text{for } m \text{ odd} \tag{55}$$

  where F is any of $\Phi_{\rm Re}, \Phi_{\rm Im}, \Pi_{\rm Re}$ or $\Pi_{\rm Im}$.

- Use the following approximate outgoing radiation conditions at $r = r_{\rm max}$:

$$\left[\frac{\partial}{\partial t}F(t,r,\theta) + \frac{\partial}{\partial r}F + \frac{F}{r}\right]_{r=r_{\rm max}} = 0 \tag{56}$$

- At the inner boundary, $r = r_{\rm min}$, use $O(\Delta r^3)$ extrapolation to determine the $r = r_{\rm min}$ values in terms of $r = r_{\rm min} + i\Delta r,\ i = 1,2,3$.

  Note: In Kerr-Schild coordinates, the location of the black hole horizon is $r_{\rm BH} = M + \sqrt{M^2 - a^2}$. *You should ensure that you always choose $r_{\rm min} \le r_{\rm BH}$, and you may find it useful to experiment with the placement of the inner boundary.*

- Use KO dissipation in both the radial and angular directions, as is done in `wave2d_sph`.
- For initial data, use the same type of modulated-gaussian pulse that is used in `wave2d_sph`, for $\Phi_{\rm Re}$, except multiply the gaussian by a factor of $\sin^2\theta$ rather than $\sin^m\theta$. Set the initial values of $\Phi_{\rm Im}$, $\Pi_{\rm Re}$ and $\Pi_{\rm Im}$ to zero. (This will generate a time-symmetric evolution.)
- Code independent residual evaluators for the evolution equations for $\Pi_{\rm Re}$ and $\Pi_{\rm Im}$.
- You may find it beneficial to use a smaller Courant factor (RNPL's `lambda` parameter) than is used in the `wave2d_sph` example.
- *IMPORTANT: Consider only the case $a = 0$ (i.e. Schwarzschild) at this stage.*

2. Once you have your code de-bugged, run some basic convergence tests for short amounts of evolution time, and for $m = 0,1,2$. Suggested problem parameters include:

   - `Nr0 = 32, Nth0 = 16`
   - `lambda = 0.2`
   - `rmin = 2.0`
   - `rmax = 20.0`
   - `iter = 128`

   Use a gaussian centred at $(r,\theta) = (10.0, \pi/2)$, with widths $(\delta r, \delta\theta) = (2.0, 1.0)$.

3. Using the above parameters, investigate the long-time (several light-crossing times) behaviour of your code for any of $m = 0,1,2$. What do you observe? Is your code stable? If not, can you deduce what is causing the instability (i.e. is it an inner boundary effect, an outer boundary effect, or ... ?)

4. The following is *one* route that you can take to attempt to cure the stability problems you should have seen by carrying out the above. *You should also feel free to attempt your own solutions!!* For this part of the problem, we suggest that you make a new version of your code—at the very least, make a backup copy of your old code!

   - Introduce a compactified radial coordinate, s, defined by

   $$s(r) = \frac{r - r_{\min}}{r+1} \tag{57}$$

     Note that $s = [0, 1]$ maps into $r = [r_{\min}, \infty]$.

   - It should be straightforward to modify your old RNPL code to incorporate s using the chain rule to modify the difference operators, and, to keep your source code readable, *after an appropriate change of variable names.* (i.e. your code should use `t, s, theta` coordinates, *not* `t, r, theta`). In particular, consider defining a grid function `dsdr` $= ds(r)/dr$:

     ```
     #Transformation Jacobian
     float dsdr  on g1 {out_gf := 0}
                  .
                  .
                  .
     initialize dsdr {
        [1:Ns-1][1:Nth]  := (1+rm)/(1+r)^2;
        [Ns:Ns][1:Nth]   := 0;
     }
     ```

     Then, for example, the following difference operator definition is an $O(h^2)$ approximation to $\partial/\partial r = (ds/dr)\partial/\partial s$:

     ```
     operator DOR(f,s) := dsdr * (<0>f[1][0] - <0>f[-1][0])/(2*ds)
     ```

     Similarly, the following is an $O(h^2)$ approximation to $\partial^2/\partial r^2$:

     ```
     operator DOR(f,s,s) := dsdr*(DP(MUM(dsdr,s)*DM(f,s),s))
     ```

     where `DP(...,s)`, `DM(...,s)` and `MUM(...,s)` are defined as previously.

   - Use homogeneous Dirichlet boundary conditions on all of the field quantities at $s = 1$ ($r = \infty$). The instructors have also found that imposing Dirichlet conditions on Π_{re} and Π_{im} at the next-to-extremal radial locations (i.e. at `i = Ns - 1`) is also useful for maintaining stability.

   - Initialize your code in r-space; to do this you may find it useful to introduce a grid function `r` and initialize it suitably via the inverse, $s(r)$, of (57).

5. As much as you are able, convergence test your code, at least for short times, as you did in part 2 of this problem. Again, we suggest at this point that you restrict attention to the case $a = 0$. Unless time is very short, you should also code an independent residual evaluator, at least for the Π_{re} evolution equation.

6. *Quasinormal ringing*: Using a Gaussian pulse, and for $m = 0, 1, 2$, attempt to measure the quasinormal frequencies (QNFs) of a Schwarzschild hole ($a = 0$). Note that QNFs are complex numbers, i.e, they are characterized by both a frequency and decay rate—you should be able to *estimate* both of these quantities. One way of measuring the QNFs is to examine the absolute value of the profile of an appropriate field variable at a fixed spatial location, i.e., look at $|\Phi_{\rm re}(t, r, \theta)| = |\Phi_{\rm re}(t, r_{\rm ex}, \pi/2)|$, where we suggest the use of an "extraction" radius of $r_{\rm ex} \sim 10$ (assuming $M = 1$). You can extract and analyze such a profile using a combination of `DV` and `xvs`—see the course web page for details.

   If time permits, investigate the dependence of the QNFs (and the "extracted" profile) on the details of the initial data configuration (e.g. vary the parameters θ_0, r_0, $\delta\theta$, δr, etc. in the Gaussian, or use initial data of your own choosing).

7. *Optional:* Try running your code for a non-zero. What happens? If your code appears to be unstable for $a \neq 0$, attempt to stabilize it

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