Project 1: The Wave Equation on the Schwarzschild Background in Eddington-Finkelstein Coordinates

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1.1 Introduction

In this project, after the derivation and verification of some equations of motion and other results, you will use `RNPL`-generated finite-difference codes to study the spherically-symmetric dynamics of a massless scalar field on a Schwarzschild (black hole) *background*—i.e. the "back reaction" of the scalar field will be ignored, so the spacetime will be fixed, and completely known *a priori*. The chief physics which this model describes, and the physics on which you will focus, is the absorption and/or scattering of spherically-symmetric pulses ("S-waves") of scalar radiation which "infall" onto a black hole, and whose self-gravitation can be ignored. You will solve the problem in the 3+1 form of ingoing Eddington-Finkelstein coordinates (see e.g. MTW [1], 31.4 & Box 31.2 for a general discussion of Eddington-Finkelstein coordinates and corresponding line-elements), which will allow you to excise the interior of the black hole simply by limiting the spatial domain of integration to the region $r \geq 2M$. Because the inner boundary, $r = 2M$, of the spatial domain is actually *null*, in principle *no special boundary conditions* are needed there for the scalar field—one simply applies the equations of motion (the covariant wave equation) up to and including $r = 2M$.

Units and Conventions We adopt MTW units (in particular $G = c = 1$) and metric conventions. Latin indices from near the start of the alphabet $(a, b, c, \cdots)$ are spacetime abstract indices (See Wald [2], "Notation and Conventions" for an explanation); Greek indices label 4-dimensional tensor components and range over the spacetime values 0, 1, 2, 3; Latin indices from near mid-alphabet $(i, j, k, \cdots)$ label 3-dimensional tensor components and take on the spatial values 1, 2, 3. The Einstein summation convention applies to both types of component index.

1.2 The Wave Equation for a General, Static, Spherically Symmetric Metric

Consider the following general 3+1 form of the static, spherically symmetric, vacuum metric (i.e. the Schwarzschild spacetime):

$$ds^2 = \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2a^2\beta\, dtdr + a^2 dr^2 + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right) \qquad (1)$$

where $\alpha \equiv \alpha(r)$, $\beta = \beta(r)$ and $a = a(r)$. In 3+1 language, α is the known as the *lapse function*, while β is the radial component of the *shift vector*, i.e. $\beta^i = (\beta, 0, 0)$, $\beta_i \equiv \gamma_{ij}\beta^j =$

$(a^2\beta, 0, 0)$, where γ_{ij} is the intrinsic metric of the 3-dimensional, spacelike hypersurfaces defined by $t = \text{const.}$. Specifically, we have $\gamma_{ij} \equiv \text{diag}(a^2, r^2, r^2 \sin^2\theta)$

We note that (1) is not *the* most general form for a static spherically-symmetric spacetime; we have chosen a so-called "areal" radial coordinate—i.e. r provides a direct measure of the area, $4\pi r^2$, of $r = \text{const.}$ spheres.

Problem 1a) Show that the characteristics (null geodesics) of (1) are given by

$$\left(\frac{dr}{dt}\right)_{\pm} = -\beta \pm \frac{\alpha}{a}. \tag{2}$$

Problem 1b) Show that, given the metric (1), the massless Klein-Gordon equation (the wave equation):

$$\nabla^a \nabla_a \phi\,(r,t) = 0 \tag{3}$$

can be written as the pair of first-order-in-time (3+1, "Hamiltonian") equations

$$\partial_t \Phi = \partial_r \left(\beta\Phi + \frac{\alpha}{a}\Pi\right), \tag{4}$$

$$\partial_t \Pi = \frac{1}{r^2}\partial_r \left(r^2 \left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right), \tag{5}$$

where

$$\Phi\,(r,t) \equiv \partial_r \phi\,, \tag{6}$$

$$\Pi\,(r,t) \equiv \frac{a}{\alpha}\left(\partial_t\phi - \beta\,\partial_r\phi\right). \tag{7}$$

Hints Use the fact that (3) can be rewritten as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\right) = 0 \tag{8}$$

where g is the determinant of the 4-metric, and where the components $g^{\mu\nu}$ of the inverse 4-metric are given by (verify)

$$g^{\mu\nu} = \begin{bmatrix} -\alpha^{-2} & \beta\alpha^{-2} & 0 & 0 \\ \beta\alpha^{-2} & a^{-2} - \beta^2\alpha^{-2} & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{bmatrix}. \tag{9}$$

1.3 The Schwarzschild Solution in Ingoing Eddington-Finkelstein Coordinates

Now consider the usual Schwarzschild form of (1), and bear in mind that, as discussed in the introduction, we will be restricting attention to $r \geq 2M$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,. \tag{10}$$

If we now define the "Regge-Wheeler tortoise coordinate", $r_\star$,

$$r_\star \equiv r + 2M \ln\left(\frac{r}{2M} - 1\right) \,, \tag{11}$$

then u and v defined by

$$u = t - r_\star \,, \tag{12}$$

$$v = t + r_\star \,. \tag{13}$$

are outgoing (u) and ingoing (v) null coordinates.

Problem 1c) Show that if we adopt a timelike coordinate, $\tilde{t}$, based on the *ingoing* null coordinate, v, as follows

$$\tilde{t} = v - r = t + 2M \ln\left(\frac{r}{2M} - 1\right) \,, \tag{14}$$

then the Schwarzschild metric (10) takes the *ingoing Eddington-Finkelstein* (IEF) form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tilde{t}^2 + \frac{4M}{r} d\tilde{t} dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \,. \tag{15}$$

Note that this form differs from the usually quoted IEF metric—for example, from MTW, Box 31.2, equation (2):

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2 dv dr + r^2 d\Omega^2 \,, \tag{16}$$

in that, in (15) we adopt a *timelike* coordinate, $\tilde{t}$, rather than the null coordinate, v. Relative to the original Schwarzschild form (10), we can summarize the IEF coordinates as follows:

- We maintain an *areal* radial coordinate, r—i.e. r continues to provide a direct measure of proper surface area.
- We choose our time coordinate, $\tilde{t}$, so that the *ingoing* tangent vector:

$$\left(\frac{\partial}{\partial \tilde{t}}\right)^a - \left(\frac{\partial}{\partial r}\right)^a \,,$$

  is null.

Observe the key property of IEF coordinates—namely that, as is evident from (15), all metric components, $g_{\mu\nu}$ are perfectly well behaved, both on the horizon of the black hole, $r = 2M$, and in the exterior region, $r > 2M$.

Let t now and in the following denote IEF time, so that we have

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{4M}{r} dr dt + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \,. \tag{17}$$

Problem 1d) Show that, in terms of the general 3+1 form (1), we have

$$\alpha = \left(\frac{r}{r+2M}\right)^{1/2}, \tag{18}$$

$$a = \alpha^{-1} = \left(\frac{r}{r+2M}\right)^{-1/2}, \tag{19}$$

$$\beta = \frac{2M}{r+2M}. \tag{20}$$

1.4 Asymptotics—Radiation Boundary Conditions

As $r \to \infty$, the IEF metric (17) approaches the Minkowskii metric in spherical coordinates

$$ds^s = -dt^2 + dr^2 + r^2 d\Omega^2 . \tag{21}$$

Problem 1e) Show that in this limit, the wave equation (3) can be written

$$\partial_{tt}(r\phi) = \partial_{rr}(r\phi) . \tag{22}$$

Clearly, the outgoing solution of (22) is

$$(r\phi)(r,t) = g(t-r) , \tag{23}$$

where g is an arbitrary function of one variable. Explicitly introducing the characteristic (wave) speed, c_+, associated with the outgoing solution (we have implicitly been assuming $c_+^2 = c_-^2 = c^2 = 1$), (23) becomes

$$(r\phi)(r,t) = g(c_+ t - r) , \tag{24}$$

Now, from Problem 1a), we have that, in the curved spacetime case

$$c_+ = -\beta + \frac{\alpha}{a}, \tag{25}$$

so that, asymptotically, we should expect

$$(r\phi)(r,t) = g\left(\left(-\beta + \frac{\alpha}{a}\right)t - r\right) , \tag{26}$$

which we can also express as

$$\partial_t(r\phi) + \left(-\beta + \frac{\alpha}{a}\right)\partial_r(r\phi) = 0 . \tag{27}$$

When solving the wave equation (4)-(5) on a *finite* spatial domain $2M \le r \le R$, we can impose (27) (and other equations derived using it) as a *boundary condition* at $r = R$. Such a condition is called an *outgoing radiation boundary condition*, or, often, a *Sommerfeld condition.*

1.5 Initial Data

With the wave equation written in the first order form (4)-(5), we must supply initial conditions

$$\Phi(r,0) = \Phi_0(r)\,, \tag{28}$$
$$\Pi(r,0) = \Pi_0(r)\,, \tag{29}$$

where $\Phi_0(r)$ and $\Pi_0(r)$ are arbitrary functions. Since we are most interested in studying the scattering of pulses of scalar radiation off of, and into, the black hole, we focus attention on data which, at the initial time, is "as ingoing as possible".

Assume that the initial configuration of the scalar field itself, $\phi(r,0) = \phi_0(r)$, describes some "pulse" shape—i.e. that $\phi_0(r)$ (effectively) has compact support—such as a "Gaussian"

$$\phi_0(r;\ A, r_0, \Delta) = A \exp\left(-\left(\frac{r-r_0}{\Delta}\right)^2\right)\,, \tag{30}$$

(where A, r_0 and Δ are adjustable parameters). Further make the approximation that

$$\partial_t (r\phi)(r,0) - \partial_r (r\phi)(r,0) = 0\,. \tag{31}$$

This approximation is *exact* for an ingoing pulse as the support of the pulse $\to \infty$, and, for finite r, amounts to ignoring curvature–backscatter in attempting to set up *precisely* ingoing initial data.

Problem 1f) From (28), (29) and (31), derive initial conditions, $\Phi_0(r)$ and $\Pi_0(r)$, in terms of $\phi_0(r)$ and $d\phi_0(r)/dr$.

1.6 A Conserved Mass for the Model

Recall that the stress energy tensor, T_{ab}, for the scalar field satisfying the wave equation (3) is

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi\,. \tag{32}$$

Consider a foliation, Σ_t, of spacetime with associated unit normal field n^a. Assume, as is the current case, that the spacetime has a timelike Killing vector field, t^a. Then we can define a energy-momentum 4-vector, J^a

$$J^a \equiv T^{ab}t_b\,, \tag{33}$$

which is conserved

$$\nabla_a J^a = \nabla_a\left(T^{ab}t_b\right) = \left(\nabla_a T^{ab}\right)t_b + T^{ab}\nabla_a t_b = \left(\nabla_a T^{ab}\right)t_b + T^{ab}\nabla_{(a}t_{b)} = 0\,, \tag{34}$$

by virtue of the conservation of T^{ab} and Killing's equations. We integrate $\nabla_a J^a$ over a spacetime volume and apply Gauss's theorem (see Wald B.2):

$$\int_V \nabla_a J^a = \int_{\partial V} J^a N_a = 0\,, \tag{35}$$

where ∂V is the (three-dimensional) boundary of the integration region, N^a is the normal vector to ∂V, and both integrals are taken with respect to the natural volume elements on the respective manifolds. If we now take our "Gaussian pillbox" to be the region bounded by any two hypersurfaces, Σ_t, $\Sigma_{t'}$, then assuming that $J^a N_a \to 0$ at spatial infinity (the timelike part of the pillbox), we have

$$\int_{\Sigma_t} J^a n_a - \int_{\Sigma_{t'}} J^a n_a = 0\,, \tag{36}$$

or

$$m_\infty = \int_{\Sigma_t} J^a n_a = \text{constant}\,. \tag{37}$$

i.e. m_∞ is our conserved mass.

Using our current 3+1 metric (1) we have

$$m_\infty = \int T^{\mu\nu} t_\nu n_\mu \, d\Sigma = \int T^\mu{}_\nu t^\nu n_\mu \, d\Sigma = \int (-\alpha T^t{}_t)(ar^2 \sin^2\theta)\, dr d\theta d\phi = 4\pi \int -r^2 \alpha a T^t{}_t \, dr \tag{38}$$

where we have used $t^\mu = (1, 0, 0, 0)$ and $n_\mu = (-\alpha, 0, 0, 0)$. For the purposes of monitoring our calculation, it is convenient to define a space- and time-dependent "mass aspect" function, $m(r,t)$, via

$$m\,(r,t) = \int_{2M}^{r} \frac{dm}{d\tilde{r}}\,(r,t)\, d\tilde{r}\,, \tag{39}$$

$$\frac{dm}{dr} \equiv -4\pi r^2 \alpha a T^t{}_t\,. \tag{40}$$

Problem 1g) Verify the following expression for dm/dr:

$$\frac{dm}{dr} = 4\pi r^2 \left(\frac{\alpha}{2a} \left(\Phi^2 + \Pi^2 \right) + \beta \Phi \Pi \right)\,. \tag{41}$$

1.7 Solution of the Equations of Motion

Problem 1h) Write an RNPL program to solve the wave equation

$$\begin{aligned}
\partial_t \Phi &= \partial_r \left(\beta \Phi + \frac{\alpha}{a} \Pi \right)\,, \\
\partial_t \Pi &= \frac{1}{r^2} \partial_r \left(r^2 \left(\beta \Pi + \frac{\alpha}{a} \Phi \right) \right)\,, \\
\Phi\,(r,t) &\equiv \partial_r \phi\,, \\
\Pi\,(r,t) &\equiv \frac{a}{\alpha} \left(\partial_t \phi - \beta\, \partial_r \phi \right)\,,
\end{aligned}$$

on the Schwarzschild background, and in IEF coordinates

$$
\begin{aligned}
\alpha &= \left(\frac{r}{r+2M}\right)^{1/2}, \\
a &= \alpha^{-1} = \left(\frac{r}{r+2M}\right)^{-1/2}, \\
\beta &= \frac{2M}{r+2M}.
\end{aligned}
$$

Take as the solution domain

$$
2M \le r \le R \qquad 0 \le t \le T\,, \tag{42}
$$

and use ingoing initial data and outgoing radiation boundary conditions as discussed above. It will be up to you to choose a value (or values) of T appropriate to the dyamics of the particular evolutions that you consider.

It is recommended that you first quickly work through the on-line tutorial *Solving a Simple 1d Wave Equation with* `RNPL` [4] available via the Computational Lab web page, and then use the `RNPL` code from that example as a template for your work. In particular, as in `w1dcn` your solution should use a Crank-Nicholson difference scheme, with $O(h^2)$ centred FDAs for the spatial derivatives in the interior of the domain, and $O(h^2)$ forwards and backwards approximations, as appropriate, for spatial derivatives at the domain boundaries.

Once you have your program implemented and thoroughly tested (including convergence testing, as described in [4]), extend your code so that it computes dm/dr and m defined by (41) and (39) respectively. In doing this, you may wish to refer to the `w1dcnm` example, also documented on-line via the Lab web page [3], which illustrates the incorporation of external Fortran code into RNPL.

When you are satisfied that your code is working properly, set $M = 1$, $R = 100$, $A = 1$ and $r_0 = 50$. Then compute what values of Δ result in 25%, 50% and 75% absorption, respectively, of the total initial mass of the scalar pulse (you should determine the values of Δ to about 10% accuracy or so).

If time permits, you may wish to make a more systematic survey of parameter space (using Δ as the parameter), and make a plot using `gnuplot` or `sm` (supermongo) showing the fractional absorption of pulse energy as a function of Δ.

References

[1] C.W. Misner, K.S. Thorne, J.A. Wheeler, "Gravitation", W.H. Freeman, San Francisco, 1973.

[2] R.M. Wald, "General Relativity", The University of Chicago Press, Chicago, 1984.

[3] *Graduate Summer School on General Relativistic Hydrodynamics: Computational Lab Page*
`http://laplace.physics.ubc.ca/~matt/grhydro/`

[4] *Solving a Simple Wave Equation with* RNPL
http://laplace.physics.ubc.ca/∼matt/grhydro/w1dcn.html