III. Gravitational Waves From Rotating Stars

A. Gravitational Waves from Bumps on Rotating Neutron Stars

The simplest source we consider is a neutron star with a bump on it. Rotational speeds of observed neutron stars are no larger than $c/3$, and even at $\Omega = \Omega_K$, the speed of the equator is less than $c/3$. The gravitational-wave energy is then likely to be dominated by mass quadrupole radiation:

Radiated energy

The linearized equations, in a transverse gauge (deDonder gauge), $\nabla_\beta \bar{h}^{\alpha\beta} = 0$, $\quad \bar{h}_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h$, have the form of a flat-space wave equation,

$$\Box \bar{h}_{\alpha\beta} = -16\pi T_{\alpha\beta}.$$

One way to find the energy radiated in gravitational waves is to write an action for this linearized field equation and deduce from it an energy-momentum tensor for the linearized gravitational field $h_{\alpha\beta}$, regarding it a field on Minkowski space. Varying the action

$$K\int \nabla_\gamma \bar{h}_{\alpha\beta}\nabla^\gamma \bar{h}^{\alpha\beta} d^4x$$

with respect to $h_{\alpha\beta}$ yields $-2K\Box\bar{h}_{\alpha\beta}$. Using the definition of $T_{\alpha\beta}$ as the variational derivative of the matter action I_M with respect to the metric,

$$T_{\alpha\beta} := 2\frac{\delta}{\delta g_{\alpha\beta}} I_M,$$

we see that the field equation is obtained (for a Minkowski space background) by varying the action

$$I = I_G + I_M = \frac{1}{64\pi}\int \nabla_\gamma \bar{h}_{\alpha\beta}\nabla^\gamma \bar{h}^{\alpha\beta} d^4x + I_M$$

with respect to $h_{\alpha\beta}$, with $g_{\alpha\beta}$ in I_M replaced by its first-order form, $\eta_{\alpha\beta} + h_{\alpha\beta}$. The stress energy tensor associated with the action I_G is

$$T_{\mathrm{G}\alpha\beta} = \frac{1}{32\pi}\nabla_\alpha \bar{h}_{\gamma\delta}\nabla_\beta \bar{h}^{\gamma\delta}.$$

The 4-momentum of gravitational waves associated with a timelike Killing vector t^α of Minkowski space is $T_{G\alpha\beta}t^\beta$, implying a rate of energy loss across a sphere S

$$\frac{dE}{dt} = \int T_{G\alpha\beta}t^\beta dS_\alpha.$$

In particular, the flux of energy across an r = constant sphere lying in a $t =$ constant hypersurface orthogonal to t^α is

$$\frac{dE}{dt} = \frac{c^4}{G}\frac{1}{32\pi}\int \partial_t \bar{h}_{\beta\gamma}\partial_r \bar{h}^{\beta\gamma} r^2 d\Omega. \tag{3.1}$$

Similarly, associated with a rotational Killing vector ϕ^α of the flat background is the angular momentum current $T_{G\alpha\beta}\phi^\beta$, whose flux across an r = constant sphere is

$$\frac{dJ}{dt} = \frac{c^4}{G}\frac{1}{32\pi}\int \partial_\phi \bar{h}_{\beta\gamma}\partial_r \bar{h}^{\beta\gamma} r^2 d\Omega. \tag{3.2}$$

For periodic motion with time dependence and angular dependence given by $\mathrm{Re}\ \exp[i(m\phi - \sigma t)]$, we immediately have

$$\left\langle \frac{dJ}{dt} \right\rangle = \frac{m}{\sigma} \left\langle \frac{dE}{dt} \right\rangle \tag{3.3}$$

For slow motion, mass quadrupole radiation is dominant, leading to the familiar quadrupole formula,

$$\frac{dE}{dt} = \frac{c^4}{G}\frac{1}{5}\langle \dddot{I}_{ab}\dddot{I}^{ab}\rangle, \tag{3.4}$$

where $I_{ab} = \int \delta\rho\, r_a r_b d^3V$.

More generally, from the energy flux equation and from a multipole expansion of the solution

$$\bar{h}_{\alpha\beta} = \frac{1}{4\pi}\int \left.\frac{T_{\alpha\beta}(t, \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\right|_{\mathrm{ret}} dV'$$

one finds energy the energy radiated by the lth mass multipole,

$$D_{lm} = K\ \mathrm{Re}\int \rho r^l Y^*_{lm}\, dV, \tag{3.5}$$

is

$$\frac{dE}{dt} = k\left(\frac{d^{l+1}}{dt^{l+1}}D_{lm}\right)^2, \tag{3.6}$$

(A standard reference for multipole radiation:
Thorne, Rev. Mod. Phys, **52**, 299, 1980.)
Each time derivative brings in an additional power of v; reinstating c and G, we have for k the dimensionful form G/c^{2l+1}, leading to $\dot{E} \sim E\Omega(\frac{v}{c})^{(2l+1)}$, where E is the system's energy, of order GM^2/R.) It is for this reason that, for v/c small, the lowest nonvanishing multipole dominates.
Because the lth current multipole,

$$J_{lm} = K \ \mathrm{Re} \int \rho r^l \boldsymbol{v} \cdot \hat{\boldsymbol{r}} \times \nabla Y^*_{lm} dV$$

is smaller by a factor of v^2 than the corresponding mass multipole, the energy radiated by the lth current multipole,

$$\frac{dE}{dt} = k\left(\frac{d^{l+1}}{dt^{l+1}}J_{lm}\right)^2,$$

is smaller than the corresponding mass-multipole radiation by v^2/c^2.

Quadrupole radiation from a bump

A bump can be described by a change $\delta\rho$ in the star's density. If we denote the $l = m = 2$ quadrupole by

$$Q \equiv D_{22} = \mathrm{Re} \int \delta\rho \, Y_{22}(r) r^2 dV, \tag{3.7}$$

the quadrupole formula (3.4) for this perturbation is equivalent to the $l = m = 2$ case of Eq. (3.6)

$$\frac{dE}{dt} = \frac{4\pi}{75} \dddot{Q}^2.$$

(To find the constant, use $Y_{22} = (15/32\pi)^{1/2}(x+iy)^2/r^2$ to obtain $\langle I_{ab} I^{ab} \rangle = (4\pi/15) Q_{22}^2$.)

Now perturbations with with $l = m = 2$ have frequency

$$\omega = 2\Omega,$$

because Y_{22} is invariant under a rotation by π about the star's rotation axis. Then

$$\frac{dE}{dt} = \frac{G}{c^5}\frac{256\pi}{75}\,\Omega^6 Q_{22}^2. \tag{3.8}$$

Angular momentum balance and wave amplitude.

From Eqs. (3.3) and (3.8), the rate of loss of angular momentum is

$$\frac{dJ}{dt} = \frac{2}{\omega}\frac{dE}{dt} = \frac{G}{c^5}\frac{256\pi}{75}\frac{\Omega^5 Q_{22}^2}{c^5}. \tag{3.9}$$

There may be accreting neutron stars (perhaps a class of low-mass X-ray binaries, LMXBs) that are spun up by accretion until the angular momentum radiated in gravitational waves is as great as the angular momentum gained in accretion. As mentioned in the next lecture, Wagoner first worked out this balance for a gravitational-wave driven (CFS) instability. Here we consider the simpler case where the radiation is due to a fixed bump. In LMXBs, the rate of accretion is close to the Eddington limit, where radiation pressure from infalling matter is as great as the gravitational attraction on that accreting matter. The

Eddington limit for neutron stars is $\dot{M} \approx 2 \times 10^{-8} M_\odot/\mathrm{yr}$; the corresponding rate at which angular momentum is deposited is then

$$\frac{dJ}{dt} \approx \dot{M} v R \approx \dot{M}(GMR)^{1/2},$$

implying $\dot{J} = 2 \times 10^{34}$ erg, for $M = 1.4 M_\odot$, $R = 10\mathrm{km}$, $\nu \equiv \Omega/2\pi = 300\mathrm{Hz}$ (a typical frequency for a neutron star in an LMXB). Equating this to the rate (3.9) of angular momentum loss, we obtain (Ushomirsky, Cutler, and Bildsten) a size of the bump that is needed, namely

$$Q = 1.6 \times 10^{38}\ \mathrm{g\ cm}^2 \left(\frac{M}{1.4 M_\odot} \frac{R}{10\ \mathrm{km}}\right)^{1/4} \left(\frac{\dot{M}}{2 \times 10^{-8} \frac{M_\odot}{\mathrm{yr}}}\right)^{1/2} \left(\frac{300\mathrm{Hz}}{\nu}\right)^{5/2} \tag{3.10}$$

The corresponding amplitude h of gravitational waves at a distance d can be obtained from Eq. (3.2):

$$\frac{dJ}{dt} \sim \frac{c^3}{G} \Omega d^2 h^2 \ \Rightarrow \ h \sim 10^{-26},$$

for a source at the distance of the galactic center ($d \sim 10\text{kpc}$). With accretion not quite at the Eddington limit and with constants kept, a better estimate is $h \sim 10^{-27}$. This is still an interesting value: "Prior accurate knowledge of the position on the sky and orbital periods of many of these X-ray binaries will allow for deep searches with the suite of laser-interferometric gravitational wave detectors."

To produce the necessary displacement, Ushomirsky et al., following an earlier paper by Bildsten, show that a temperature gradient from accretion can lead to a significant deformation of the inner crust, with heat from accretion driving nuclear reactions: Heating crustal matter decreases its density; its original nuclear composition is no longer the equilibrium composition, and nuclear reactions produce a new equilibrium with a composition altered most where the temperature is greatest. The anisotropic composition leads to an anisotropic density distribution, in which layers of the crust have been slightly displaced. Because stresses can locally deviate from isotropy only because of a crustal lattice, the size of the displacement is limited by the maximum shear strain, the maximum value of $|\nabla\xi|$, where ξ is the displacement. A shear strain of order 10^{-2} is needed to limit neutron star rotation and produce waves with the amplitude estimated above.

But Thompson and Duncan's model for soft gamma repeaters (SGRs) has a crust cracking at strain $\sim 10^{-3}$, and earlier estimates were still smaller.