Astrophysical Sources of Gravitational Waves

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Anticipated Sources

I. Rotating Neutron Stars

A. Rotating Bumps

B. Unstable Nonaxisymmetric Modes

II. Compact Binaries

A. Neutron-Star Binaries

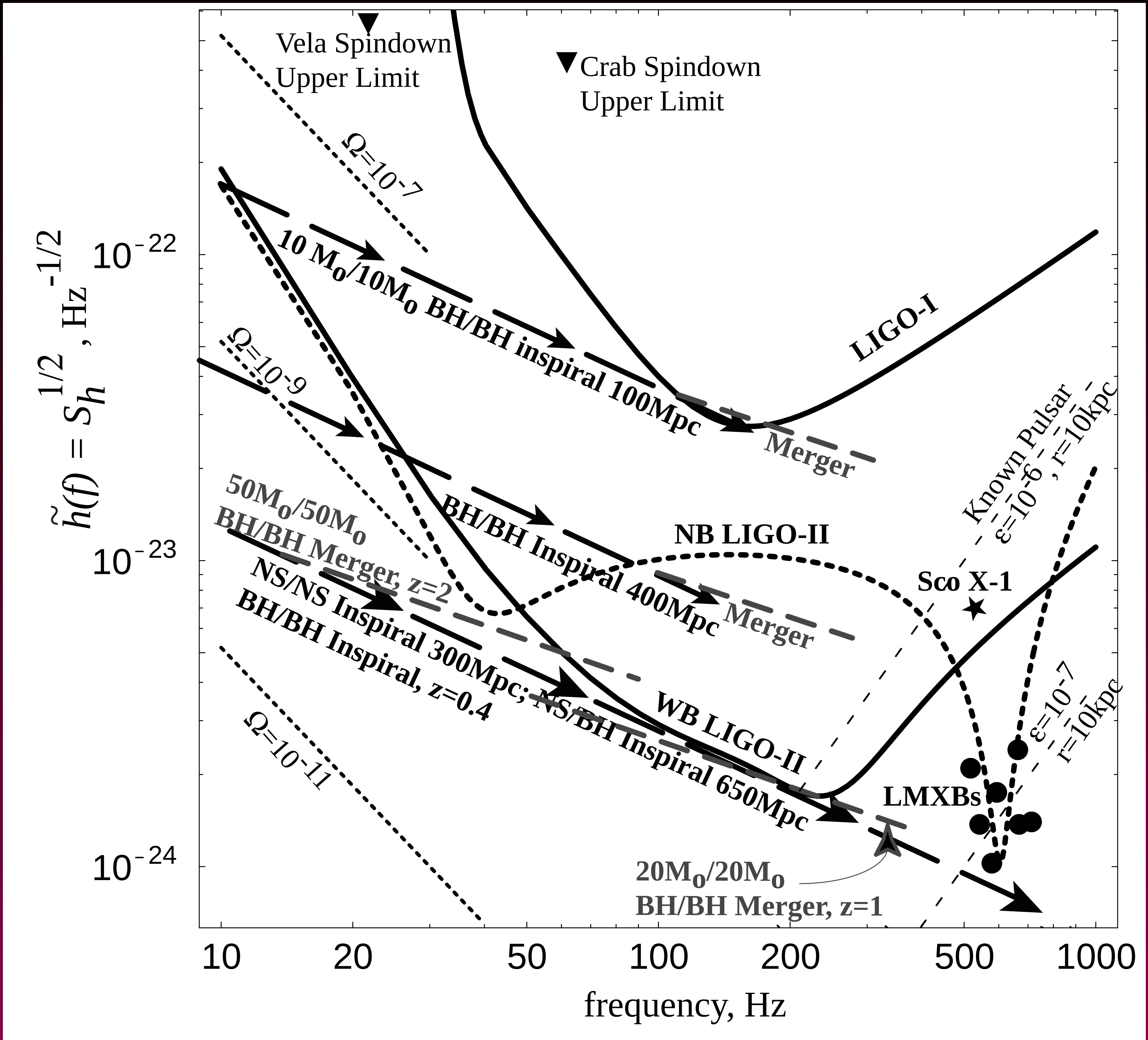
B. Stellar-Size Black-Hole Binaries

C. NS-BH Binaries

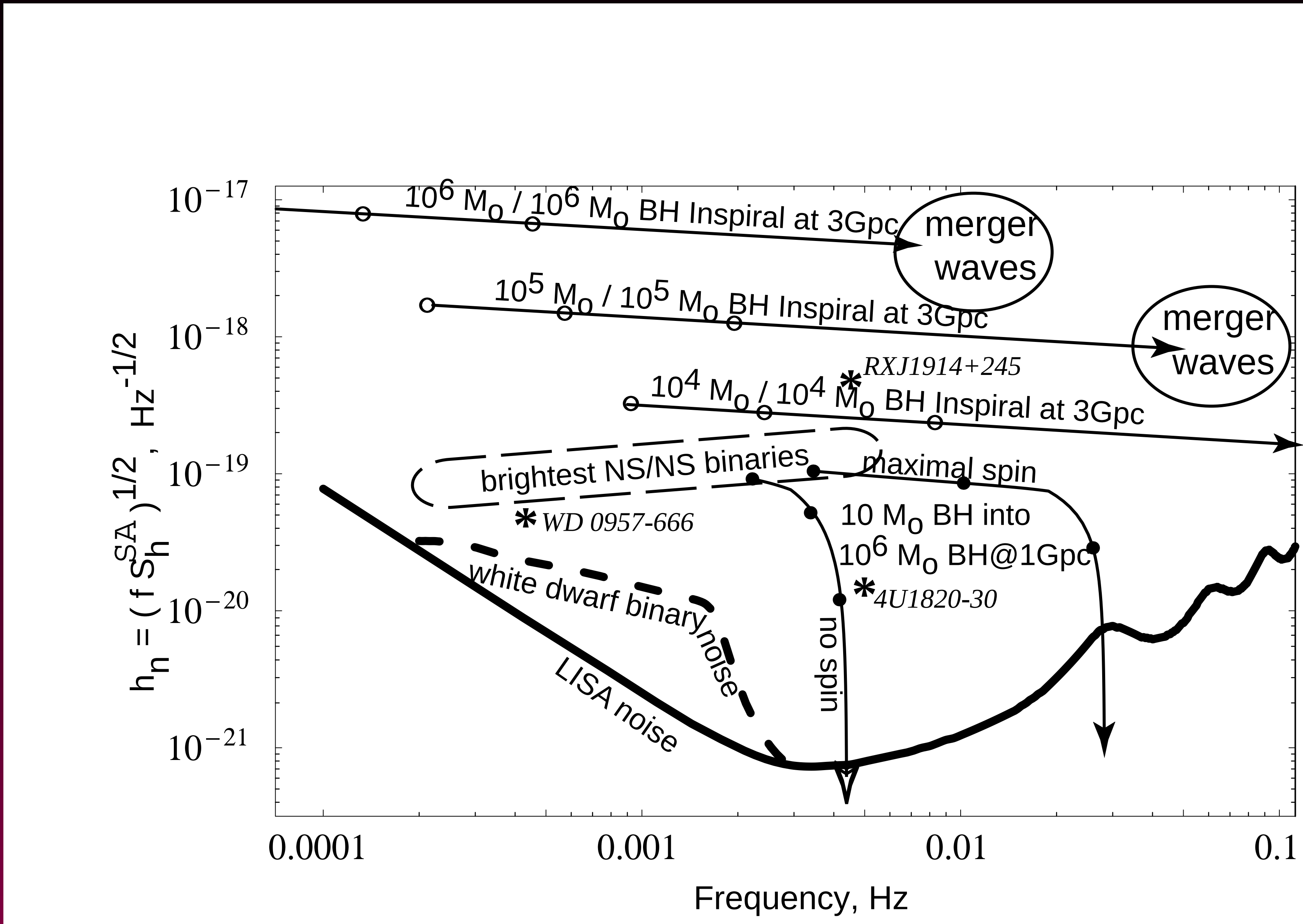
D. Binaries with a Supermassive Black Hole

III. Core Collapse in Supernova or Hypernova

IV. Stochastic Background



Vela Spindown
Upper Limit
Crab Spindown
Upper Limit
Ω=10-7
Ω=10-9
Ω=10-11
10 Mo/10Mo BH/BH inspiral 100Mpc
Merger
LIGO-I
BH/BH Inspiral 400Mpc
Merger
NB LIGO-II
50Mo/50Mo
BH/BH Merger, z=2
NS/NS Inspiral 300Mpc; NS/BH Inspiral 650Mpc
BH/BH Inspiral, z=0.4
WB LIGO-II
Known Pulsar
ε=10-6, r=10kpc
ε=10-7
r=10kpc
Sco X-1
LMXBs
20Mo/20Mo
BH/BH Merger, z=1
10^{-22}
10^{-23}
10^{-24}
$\tilde{h}(f) = S_h^{1/2}$, $Hz^{-1/2}$
10
20
50
100
200
500
1000
frequency, Hz



Cutler& Thorne

Outline

Each topic, I, II, III, and IV, below, is posted as a separate file. Because the breaks between the lectures themselves were dictated by time, lecture breaks do not in general coincide with the end of a topic.

I. General Relativistic Perfect Fluids

A. Definition of Perfect Fluid: Local Isotropy

B. The Einstein-Euler System

C. Barotropic Flows: Enthalpy, the Bernoulli Equation, Injection Energy, and Conservation of Circulation

II. Relativistic Stars

A. Spherical Stars

B. Rotating Relativistic Stars

III. Gravitational Waves from Rotating Stars

A. Rotating Bumps on Neutron Stars

B. Waves From Dynamical Instability of A Rapidly Rotating, Collapsing Core

C. Gravitational Wave Driven Instability of Nascent Neutron Stars and of Old Stars Spun Up by Accretion

IV. Compact Binaries: Quasistationary Equilibria

A. Data Sets and Full Solutions with Helical Symmetry

B. First Law of Thermodynamics for Binary Black Holes and Neutron Stars; Turning Point Instability; Locating the ISCO

I. General Relativistic Perfect Fluids

Note: In these talks,

ε= energy density,
ρ= baryon rest-mass density,
n = baryon number density.

Then $\rho = m_B n$, where m_B is the rest mass per baryon.

Energy-momentum tensor

A perfect fluid is a model for a large assembly of particles in which a continuous energy density ε can reasonably describe the macroscopic distribution of mass. One assumes that the microscopic particles collide frequently enough that their mean free path is short compared with the scale on which the density changes, that the collisions enforce a local thermodynamic equilibrium. In particular, one assumes that a mean velocity field

u^α and a mean energy-momentum tensor $T^{\alpha\beta}$ can be defined in boxes – fluid elements – small compared to the macroscopic length scale but large compared to the mean free path. And on scales large compared to the size of the fluid elements, the 4-velocity and thermodynamic quantities can be accurately described by continuous fields. An observer moving with the average velocity u^α of the fluid will see the collisions randomly distribute the nearby particle velocities so that the particles will look locally isotropic.

Because a comoving observer sees an isotropic distribution of particles, the components of the fluid's energy momentum tensor in his frame must have no preferred direction: $T^{\alpha\beta}u_\beta$ must be invariant under rotations that fix u^α. Denote by

$$q^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta \tag{1.1}$$

the projection operator orthogonal to u^α.

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Similarly, the symmetric tracefree tensor
$^{3}T^{\alpha\beta} - \frac{1}{3}q^{\alpha\beta}\,{}^{3}T \equiv q^{\alpha}_{\gamma} q^{\beta}_{\delta} T^{\gamma\delta} - \frac{1}{3}q^{\alpha\beta} q_{\gamma\delta} T^{\gamma\delta}$
transforms as a $j = 2$ representation of the rotation group and can be invariant only if it vanishes.

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It follows that the only nonzero parts of $T^{\alpha\beta}$ are the rotational scalars

$$\varepsilon \equiv T^{\alpha\beta} u_{\alpha} u_{\beta} \tag{1.2}$$

and

$$P \equiv \frac{1}{3} q_{\gamma\delta} T^{\gamma\delta}. \tag{1.3}$$

More concretely, in an orthonormal frame with $\mathbf{e_0}$ along u^α,
T^{0i} and $T^{ij} - \frac{1}{3}\delta^{ij}T^k_k$ must vanish,
implying that $T^{\alpha\beta}$ has components

$$\|T^{\mu\nu}\| = \left\| \begin{matrix} \varepsilon & & & \\ & P & & \\ & & P & \\ & & & P \end{matrix} \right\|.$$

Summary:
The condition of local isotropy suffices to define a perfect fluid, by enforcing an energy-momentum tensor of the form

$$T^{\alpha\beta} = \varepsilon u^\alpha u^\beta + P q^{\alpha\beta} = (\varepsilon + P) u^\alpha u^\beta + P g^{\alpha\beta}. \qquad (1.4)$$

Departures from a perfect fluid

In neutron stars, departures from perfect fluid equilibrium due to a solid crust are expected to be $\sim 10^{-3}$ or smaller, corresponding to the maximum strain that an electromagnetic lattice can support.

On a submillimeter scale, superfluid neutrons and protons in the interior of a neutron star have velocity fields that are curl-free outside a set of quantized vortices. On larger scales, however, a single, averaged, velocity field u^α accurately describes a neutron star (Baym and Chandler 1983; Sonin 1987; Mendell and Lindblom 1991). Although the approximation of uniform rotation is consequently invalid on scales shorter than 1 cm, the error in computing the structure of the star on larger scales is negligible. In particular, with $T^{\alpha\beta}$ approximated by a value $< T^{\alpha\beta} >$ averaged over several cm, the error in computing the metric is of order

$\delta g_{\alpha\beta} \sim (\frac{1\,\mathrm{cm}}{R})^2 \sim 10^{-11}$.

For equilibria, these are the main corrections. For dynamical evolutions – oscillations, instabilities, collapse, and binary inspiral, one must worry about the microphysics governing, for example viscosity, heat flow, magnetic fields, superfluid modes, and turbulence.

B. The Einstein-Euler Equations

A perfect-fluid spacetime is a spacetime M, g whose source is a perfect fluid. That is, the metric satisfies

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

with $T_{\alpha\beta}$ a perfect-fluid energy-momentum tensor. The Bianchi identities imply

$$\nabla_\beta T^{\alpha\beta} = 0,$$

and this equation, together with an equation of state, determines the motion of the fluid.

The projection of the equation $\nabla_\beta T^{\alpha\beta} = 0$ along u^α yields an energy conservation law, while the projection orthogonal to u^α is the relativistic Euler equation. For an intuitive understanding of these equations, it is helpful to look first at conservation of baryons.

Conservation of baryons

The baryon mass M_0 of a fluid element is conserved by the motion of the fluid. The proper volume of a fluid element is the volume V of a slice $\perp u^\alpha$ through the history of the fluid element; and conservation of baryons can be written in the form $0 = \Delta M_0 = \Delta(\rho V)$. The fractional change in V in a proper time $\Delta\tau$ is given by the 3-dimensional divergence of the velocity, in the subspace orthogonal to u^α:

$$\frac{\Delta V}{V} = q^{\alpha\beta}\nabla_\alpha u_\beta \Delta\tau. \tag{1.5}$$

Because $u^\beta u_\beta = -1$, we have $u^\beta\nabla_\alpha u_\beta = \frac{1}{2}\nabla_\alpha(u_\beta u^\beta) = 0$, implying

$$q^{\alpha\beta}\nabla_\alpha u_\beta = \nabla_\beta u^\beta.$$

With $u^\alpha\nabla_\alpha\rho = \dfrac{d}{d\tau}\rho$, conservation of baryons takes the form

$$0 = \frac{(\Delta\rho V)}{V} = \Delta\rho + \rho\frac{\Delta V}{V} = (u^\alpha\nabla_\alpha\rho + \rho\nabla_\alpha u^\alpha)\Delta\tau,$$

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or

$$\nabla_\alpha(\rho u^\alpha) = 0. \tag{1.6}$$

A more formal derivation is given below, in part to introduce a perturbation formalism that one needs to discuss the Hamiltonian formalism, stellar oscillations and stability, the virial theorem, and thermodynamics of neutron stars and black holes.

Conservation of energy

$u_\alpha \nabla_\beta T^{\alpha\beta} = 0$

$$\begin{aligned} 0 &= u_\alpha \nabla_\beta T^{\alpha\beta} = u_\alpha \nabla_\beta [\varepsilon u^\alpha u^\beta + P q^{\alpha\beta}] \\ &= -\nabla_\beta(\varepsilon u^\beta) + P u_\alpha \nabla_\beta (g^{\alpha\beta} + u^\alpha u^\beta) \\ &= -\nabla_\beta(\varepsilon u^\beta) - P \nabla \cdot u \end{aligned}$$

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The equation means that the mass of a fluid element decreases by the work,

$$P\, dV = -PV\, \nabla \cdot u\, d\tau,$$

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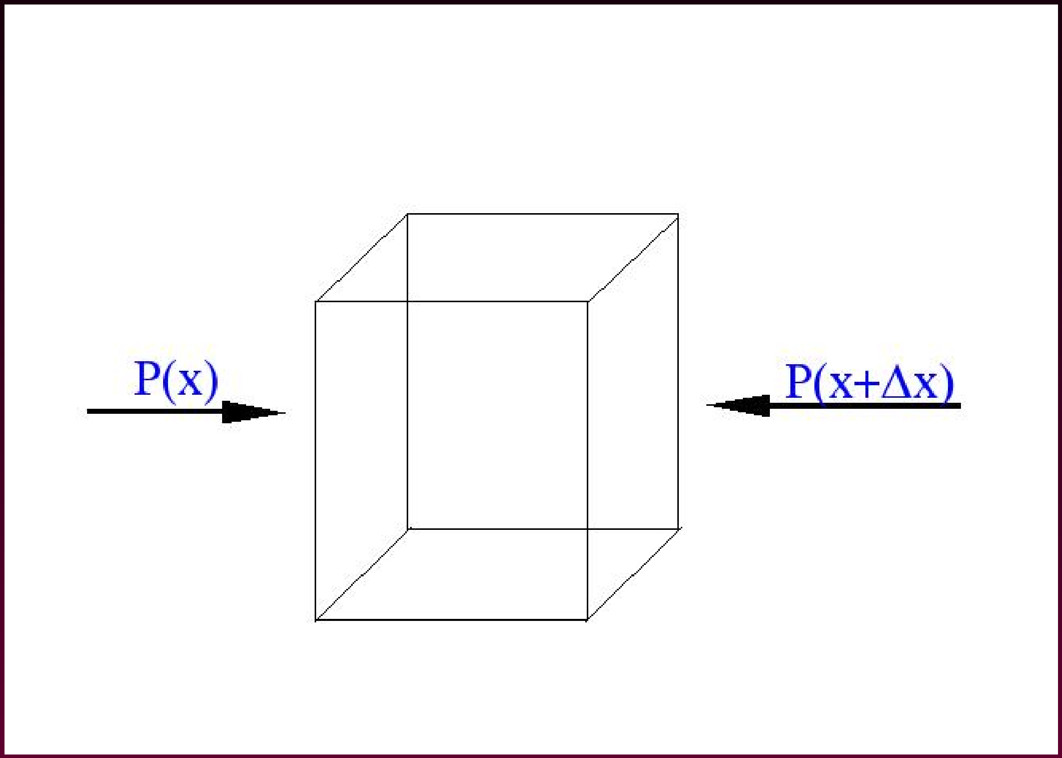
Relativistic Euler equation:

$$q^{\alpha}{}_{\gamma}\nabla_{\beta}T^{\beta\gamma} = 0$$



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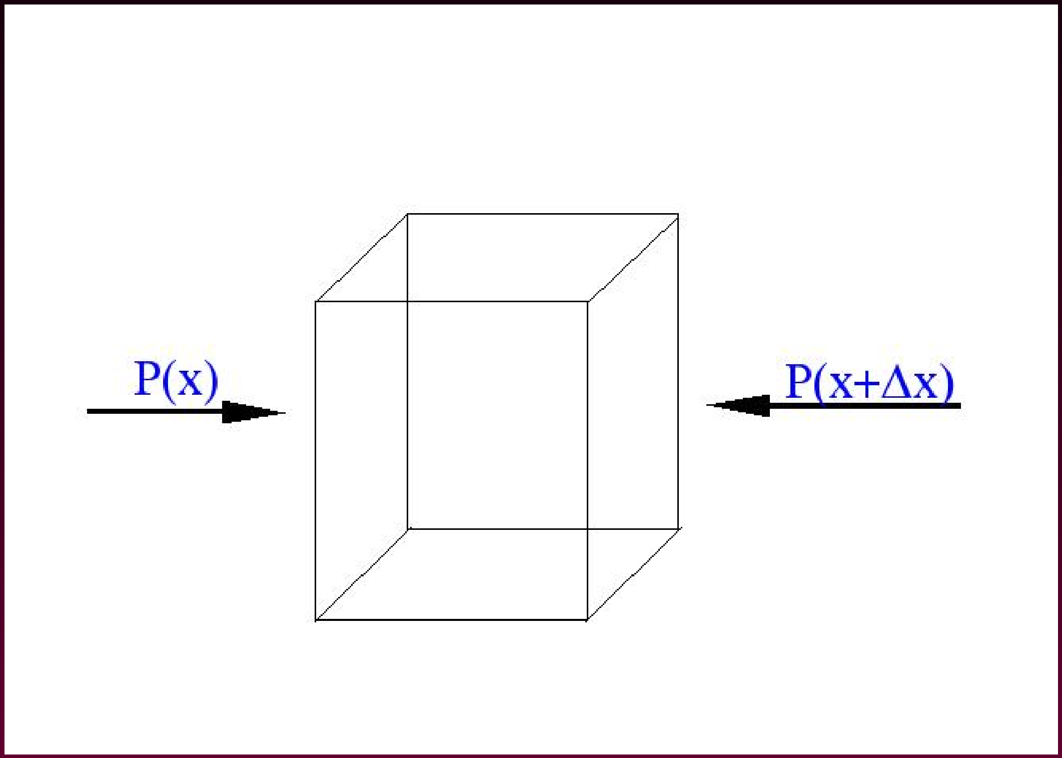

$$\begin{aligned}0 &= q^{\alpha}{}_{\gamma}\nabla_{\beta}[\varepsilon u^{\beta}u^{\gamma}+Pq^{\beta\gamma}]\\ &= q^{\alpha}{}_{\gamma}\varepsilon u^{\beta}\nabla_{\beta}u^{\gamma}+q^{\alpha\beta}\nabla_{\beta}P+q^{\alpha}{}_{\gamma}P\nabla_{\beta}(u^{\beta}u^{\gamma})\\ &= \varepsilon u^{\beta}\nabla_{\beta}u^{\alpha}+q^{\alpha\beta}\nabla_{\beta}P+Pu^{\beta}\nabla_{\beta}u^{\alpha}\end{aligned}$$

$$(\varepsilon + P)u^{\beta}\nabla_{\beta}u^{\alpha} = -q^{\alpha\beta}\nabla_{\beta}P. \tag{1.8}$$

Newtonian limit: Let e be a small parameter of order v/c or v_{sound}/c, whichever is larger.

$$\begin{aligned}
& u^\mu = (1, v^i) + 0\,(e^2) \\
& P/\varepsilon = 0\,(e^2) \\
& \varepsilon = \rho + 0\,(e^2) \\
(1.7) \Longrightarrow \quad & \partial_t(\rho u^t) + \partial_i(\rho u^i) = -P(\partial_t u^t + \partial_i u^i) \\
& \partial_t \rho + \partial_i(\rho v^i) = 0 + 0\,(e^2) \\
(1.8) \Longrightarrow \quad & \rho u^\mu \nabla_\mu u^i = -\nabla^i P \\
& \rho(\partial_t + v^j \nabla_j) v_i + \rho \partial_i \Phi = -\nabla_i P.
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Relativistic energy conservation, Eq. (1.7), also implies the Bernoulli equation, expressing energy conservation in a Newtonian flow. We have only looked at its lowest-order form, obtaining conservation of mass at order $(v/c)^0$; to extract Newtonian energy conservation, one must keep terms at the next nonvanishing order, order v^2/c^2.

C. Barotropic flows: enthalpy, the Bernoulli equation, injection energy, and conservation of circulation

A fluid with a one-parameter EOS is called barotropic. Neutron star matter is accurately described by a one-parameter EOS because it is approximately *isentropic*: It has nearly constant (nearly zero) entropy per baryon. (There is, however, a composition gradient in neutron stars, with the density of protons and electrons ordinarily increasing outward, and this dominates a departure from a barotropic equation of state in stellar oscillations).

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The quantity

$$h = (\varepsilon + P)/\rho$$

is the enthalpy per unit rest mass. In the Newtonian limit,

$$h - 1 \longrightarrow u + P/\rho, \qquad (1.9)$$

the Newtonian specific enthalpy, with u the internal energy per unit mass.

A stationary flow is described by a spacetime with a timelike Killing vector, t^α, the generator of time-translations that leave the metric and the fluid variables fixed:

$$\pounds_t g_{\alpha\beta} = \pounds_t u^\alpha = \pounds_t \varepsilon = \pounds_t P = 0. \tag{1.10}$$

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Bernoulli's law is the Newtonian conservation of enthalpy for a stationary flow, and its relativistic form is

$$\pounds_u \left(h u_\beta t^\beta \right) = \pounds_u \left(\frac{\varepsilon + P}{\rho} u_\beta t^\beta \right) = 0. \tag{1.11}$$

To obtain Eq. (1.11), one uses the relation,

$$\frac{u^\alpha \nabla_\alpha h}{h} = \frac{u^\alpha \nabla_\alpha P}{\varepsilon + P}, \tag{1.12}$$

which itself follows from conservation of energy and baryon number: That is, from

$$\frac{u^\alpha \nabla_\alpha \varepsilon}{\varepsilon + P} = -\nabla_\alpha u^\alpha = \frac{u^\alpha \nabla_\alpha \rho}{\rho}, \tag{1.13}$$

we have,

$$u^\alpha \nabla_\alpha \left(\frac{\varepsilon + P}{\rho} \right) = \frac{1}{\rho} (u^\alpha \nabla_\alpha \varepsilon + u^\alpha \nabla_\alpha P) - \frac{\varepsilon + P}{\rho^2} u^\alpha \nabla_\alpha \rho = \frac{u^\alpha \nabla_\alpha P}{\rho} . \tag{1.14}$$

Because

$$\pounds_u u_\alpha = u^\beta \nabla_\beta u_\alpha + u_\beta \nabla_\alpha u^\beta = u^\beta \nabla_\beta u_\alpha, \tag{1.15}$$

the Euler equation, (1.8), becomes

$$\pounds_u (h u_\alpha) = -\frac{\nabla_\alpha P}{\rho} . \tag{1.16}$$

Contracting this form of the Euler equation with t^α and using Eq. 1.10, we obtain Eq. (1.11).

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The derivation holds for any Killing vector that Lie-derives the fluid variables, and, for an axisymmetric flow, yields conservation of a fluid element's angular momentum in the form,

$$£_u (h u_\beta \phi^\beta) = 0. \tag{1.17}$$

From a mathematical perspective, introducing a conserved baryon number is merely convenient. Instead of defining the specific enthalpy by $h=(\varepsilon+P)/\rho$, one can take as the definition

$$h=\exp\int_0^P\frac{dP}{\varepsilon(P,s)+P}. \tag{1.18}$$

Again one has Eq. 1.12,

$$\frac{u^\alpha\nabla_\alpha h}{h}=\frac{u^\alpha\nabla_\alpha p}{\varepsilon+p}, \tag{1.19}$$

implying the corresponding Bernoulli equation,

$$\pounds_u(hu_\beta t^\beta)=0. \tag{1.20}$$

One needs additional physics, the relations, $\varepsilon/\rho\longrightarrow 1$ and $h\longrightarrow 0$, as $P\longrightarrow 0$ for fixed s, to make the identification,

$$h=\frac{\varepsilon+P}{\rho}. \tag{1.21}$$

To order e^2 in u^α and ε,

$$u_t = 1 + \frac{1}{2}v^2 + \Phi, \qquad \varepsilon = \rho + u,$$

and the relativistic Bernoulli equation takes its Newtonian form,

$$(\partial_t + \pounds_v)(h_{\rm Newt} + \frac{1}{2}v^2 + \Phi) = 0.$$

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Hydrostatic equilibrium

A rotating star has a timelike Killing vector t^α and a rotational Killing vector ϕ^α (equivalently, $\boldsymbol{\partial}_\phi$ and $\boldsymbol{\partial}_t$). The star's 4-velocity u^α lies along a linear combination of these vectors, along the helical Killing vector

$$k^\alpha = t^\alpha + \Omega\phi^\alpha.$$

One can choose coordinates t and ϕ for which

$$t^\alpha\nabla_\alpha t = 1, \qquad \phi^\alpha\nabla_\alpha\phi = 1, \tag{1.22}$$

$$t^\alpha\nabla_\alpha\phi = 0, \qquad \phi^\alpha\nabla_\alpha t = 0. \tag{1.23}$$

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Then

$$u^\alpha = u^t(t^\alpha + \Omega\phi^\alpha) = u^t k^\alpha.$$

In the relativistic Euler equation, one can write the covariant accleration vector $u \cdot \nabla u_\alpha$ as a Lie derivative:

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Since $u^\beta\nabla_\beta P = u^t k^\beta\nabla_\beta P = 0$, we have

$$q_\alpha{}^\beta\nabla_\beta P = \nabla_\beta P \tag{1.28}$$

$$\frac{\nabla_\beta P}{\varepsilon + P} = \nabla_\beta \log h \tag{1.29}$$

Euler's equation thus has the form,

$$-\nabla_\alpha \log u^t = -\nabla_\alpha \log h \Rightarrow \nabla_\alpha \log\frac{h}{u^t} = 0;$$

with first integral

$$\frac{h}{u^t} = \mathcal{E}, \text{ constant throughout the star.} \tag{1.30}$$

This is the (first integral of) the equation of hydrostatic equilibrium for a uniformly rotating, barotropic star.

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$$\frac{h}{u^t} = \mathcal{E}, \text{ constant throughout the star.} \tag{1.30}$$

This is the (first integral of) the equation of hydrostatic equilibrium for a uniformly rotating, barotropic star.

$\mathcal{E}$ is the *injection energy per baryon*, the energy needed to lower a collection of baryons at zero temperature from infinity, expand a volume to accomodate them, add kinetic energy to match the rotation of the star, and insert them in the star.

In the numerical relativity literature the relativistic equation of hydrostatic equilibrium is often (mistakenly, I believe) called Bernoulli's law.

In Bernoulli's law,

hu_t is conserved along the fluid worldlines;

In the equation of hydrostatic equilibrium,

h/u^t is constant througout a uniformly rotating star.

The Newtonian limit of hu_t is

$h_{Newt} + \frac{1}{2}v^2 + \Phi;$

the Newtonian limit of h/u^t is

$h_{Newt} - \frac{1}{2}v^2 + \Phi.$

The flow of an isentropic fluid conserves circulation:
If one defines a relativistic vorticity $\omega_{\alpha\beta}$ by

$$\omega_{\alpha\beta} = \nabla_\alpha(hu_\beta) - \nabla_\beta(hu_\alpha), \tag{1.31}$$

the differential conservation law is the curl of the Euler equation,

$$\pounds_u \omega_{\alpha\beta} = 0. \tag{1.32}$$

The corresponding integral law is obtained as follows. Let c be a closed curve in the fluid, bounding a 2-surface Σ; and let c_τ be the curve obtained by moving each point of c a proper time τ along the fluid trajectory through that point:

$$c_\tau = \psi_\tau \circ c.$$

From the relation,

$$\pounds_u \omega_{\alpha\beta} = \nabla_\alpha \pounds_u(hu_\beta) - \nabla_\beta \pounds_u(hu_\alpha), \tag{1.33}$$

we have,

$$0 = \int_\Sigma \pounds_u \omega_{\alpha\beta} dS^{\alpha\beta} = \int_c \pounds_u (h u_\alpha) d\ell^\alpha \qquad (1.34)$$

$$= \frac{d}{d\tau}\int_c \psi_{-\tau}(h u_\alpha)\, d\ell^\alpha = \frac{d}{d\tau}\int_{c_\tau} h u_\alpha d\ell^\alpha. \qquad (1.35)$$

Stokes' theorem was used to obtain the equality on the first line; and the equality on the second line follows from the invariance of an integral under diffeos:

$$\int_{\psi_\tau c} \psi_\tau \sigma_\alpha d\ell^\alpha = \int_c \sigma_\alpha d\ell^\alpha, \text{ or } \int_c \psi_{-\tau}\sigma_\alpha d\ell^\alpha = \int_{\psi_\tau c} \sigma_\alpha d\ell^\alpha.$$

The first integral of the vorticity conservation equation is then conservation of circulation: The line integral,

$$\int_{c_\tau} h u_\alpha d\ell^\alpha = \int_{c_\tau} \frac{\varepsilon + P}{\rho} u_\alpha d\ell^\alpha, \qquad (1.36)$$

is independent of τ, conserved by the fluid flow.

The most interesting feature of this conservation law is that it is *exact in time-dependent spacetimes*, with gravitational waves carrying energy and angular momentum away from a system. In particular, oscillating stars and radiating binaries, if modeled as barotropic fluids with no viscosity or other dissipation, exactly conserve circulation.