Project 2: Critical Behaviour in Yang-Mills Collapse

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1 Introduction

This project investigates the spherically symmetric gravitational collapse of an SU(2) Yang-Mills field as described in [1, 2]. For the purposes of this project we focus on Type I critical behaviour where dispersive and collapse solutions are separated by an unstable critical solution. This critical solution is the static solution first discovered by Bartnik and McKinnon who assumed the existence of a static solution and solved the resulting ODEs via a shooting technique to find the critical solutions [2, 3].

1.1 Equations of Motion

By restricting the problem to spherical symmetry, adopting the purely magnetic ansatz of the Yang-Mills field and adopting polar/areal (PA) coordinates, a simple lagrangian for the model may be written down which is very similar to that of a simple scalar field [1].

The metric is:

$$ds^2 = -\alpha(r,t)^2 dt^2 + a(r,t)^2 dr^2 + r^2 d\Omega^2 \tag{1}$$

and the matter is described by a single scalar field $W(r,t)$ with a Lagrangian [1]:

$$L_M = -\left(\frac{g^{\mu\nu} \nabla_\mu W \nabla_\nu W}{r^2} + \frac{\left(1 - W^2\right)^2}{2r^2} \right) \tag{2}$$

The total Lagrangian density, $\mathcal{L}$, is [1]:

$$\mathcal{L} = \mathcal{L}_G + \alpha_M \mathcal{L}_M = \sqrt{-g}\left(R + \alpha_M L_M\right) \tag{3}$$

where α_M is a coupling constant of arbitrary magnitude. Defining the stress energy tensor as [1]:

$$T_{\mu\nu} = \frac{\alpha_M}{8\pi}\left(-\frac{\partial L_M}{\partial g^{\mu\nu}} + \frac{1}{2} g_{\mu\nu} L_M \right) \tag{4}$$

then extremization of the action with respect to $g^{\mu\nu}$ yields the Einstein field equations while variation with respect to the matter fields produces the equations of motion.

Defining:

$$\begin{aligned} \Phi &= W' && (5) \\ \Pi &= \frac{a}{\alpha}\dot{W} && (6) \end{aligned}$$

and varying the action with respect to W, we can immediately write:

$$\begin{aligned} \dot{\Phi} &= \dot{W}' = \left(\frac{\alpha}{a}\Pi\right)' && (7) \\ \dot{\Pi} &= \frac{\alpha a}{r^2}W(1-W^2) + (\frac{\alpha}{a}\Phi)' && (8) \end{aligned}$$

The Hamiltonian constraint, momentum constraint and polar slicing condition [4],

$$\begin{aligned} R &= 16\pi\rho && (9) \\ -4\pi j_r &= K_\theta^{\theta\prime} + \frac{(rb)'}{rb}\left(K_\theta^\theta - K_r^r\right) && (10) \\ \dot{K_\theta^\theta} &= \beta + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left(\frac{\alpha rb}{a}(rb)'\right)' + \alpha\left(KK_\theta^\theta + 4\pi\left(S_r^r - \rho\right)\right) && (11) \end{aligned}$$

require the determination of the 3+1 quantities:

$$\begin{aligned} \rho &= n^\mu n^\mu T_{\mu\nu} && (12) \\ j_i &= -n_\mu T_i^\mu && (13) \\ S_j^i &= \gamma^{i,k}S_{k,j} = \gamma^{ik}T_{kj} && (14) \end{aligned}$$

While in polar areal coordinates we have [4]:

$$\begin{aligned} \beta &= 0 && (15) \\ K &= K_r^r && (16) \\ K_\theta^\theta &= K_\phi^\phi = 0 && (17) \end{aligned}$$

Evaluating the relevant quantities (setting $\alpha_M = 4$), we find:

$$\begin{aligned} \rho &= \frac{1}{2\pi\alpha^2}\left(\frac{\dot{W}^2}{2r^2} + \frac{\alpha^2}{a^2}\frac{W'^2}{2r^2} + \frac{\alpha^2}{4r^4}\left(1-W^2\right)^2\right) && (18) \\ j_r &= -\frac{\dot{W}W'}{2\pi\alpha r^2} && (19) \\ j_\theta &= j_\phi = 0 && (20) \\ S_r^r &= \frac{1}{2\pi a^2}\left(\frac{W'^2}{2r^2} + \frac{a^2}{\alpha^2}\frac{\dot{W}^2}{2r^2} - \frac{a^2}{4}\frac{\left(1-W^2\right)^2}{r^4}\right) && (21) \end{aligned}$$

And the full set of equations may be written:

$$\dot{\Phi} = \dot{W}' = \left(\frac{\alpha}{a}\Pi\right)' \tag{22}$$

$$\dot{\Pi} = \frac{\alpha a}{r^2}W(1-W^2) + (\frac{\alpha}{a}\Phi)' \tag{23}$$

$$\frac{a'}{a} = -\frac{a^2-1}{2r} + \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}\left(1-W^2\right)^2\right) \tag{24}$$

$$\frac{\alpha'}{\alpha} = \frac{a^2-1}{2r} + \frac{1}{r}\left(\Phi^2 + \Pi^2 - \frac{a^2}{2r^2}\left(1-W^2\right)^2\right) \tag{25}$$

$$\dot{a} = \frac{2\alpha}{r^2}\Pi\Phi \tag{26}$$

In the above system of equations, the equations for α and a are constraints that are applied at every timestep. As such, equation 26 is superfluous but yields a non trivial consistency check for the system; differentiating equation 24 with respect to time and 26 with respect to position should yield identical expressions. It is of course possible to do this by hand, but a symbolic manipulation program such as Maple easily verifies the consistency of the equations.

1.2 Vacuum States and Regularity Conditions

The Yang-Mills field has exactly two vacuum states, $W(r,t) = \pm 1$ [1]. During evolution we demand that W remains in specific vacuum states at $r = 0$ and $r \to \infty$. We set $W(0,t) = W_0 = 1$ and anticipating the Bartnik-McKinnon critical solution, demand that $W(r \to \infty, t) = -1$. It can be shown that the regularity of Yang-Mills field and the geometric variables requires [1]:

$$\lim_{r\to 0} W(r,t) = 1 + r^2 W_2 + O(r^4) \tag{27}$$

$$\lim_{r\to 0} a(r,t) = 1 + r^2 a_2 + O(r^4) \tag{28}$$

$$\lim_{r\to 0} \alpha(r,t) = \alpha_0 + r^2 \alpha_2 + O(r^4) \tag{29}$$

$$\tag{30}$$

1.3 Generation of Bartnik-McKinnon Solutions

The Bartnik-McKinnon solutions are unstable static solutions characterized by the number of zero crossing the the Yang-Mills potential. Assuming a static solution, the equations become:

$$\frac{a'}{a} = -\frac{a^2-1}{2r} + \frac{1}{r}\left(\Phi^2 + \Pi^2 + \frac{a^2}{2r^2}\left(1-W^2\right)^2\right) \tag{31}$$

$$W' = \Phi \tag{32}$$

$$\Phi' = -\frac{(a^2-1)}{r}\Phi + \frac{a^2}{r^3}\left(1-W^2\right)^2\Phi - \frac{a^2}{r^2}W\left(1-W^2\right) \tag{33}$$

Recalling our regularity conditions at $r = 0$ it can be seen that the solution in the vicinity of the origin must have the form:

$$a(r,t) = 1 + 2W_2^2 r^2 \quad (34)$$
$$W(r,t) = 1 + W_2 r^2 \quad (35)$$
$$\Phi(r,t) = 2W_2 r \quad (36)$$

Where W_2 is a free parameter describing the curverature of the Yang-Mills potential at $r = 0$. A program was written which used a binary search technique to converge in on the static solutions for the Yang-Mills potential. This program was originally written in Fortran using LSODA but the limits of double precision prevented the evolution of the equations past $r \approx 5000$ (i.e. the solutions would diverge at this point no matter how refined the search became). In order to integrate the equations out further, the solution was instead computed using Maple's numerical ODE tools with a tolerance set to 1E-20 and 40 digits of precision. Figure 1 shows the first three Bartnik-McKinnon solutions.

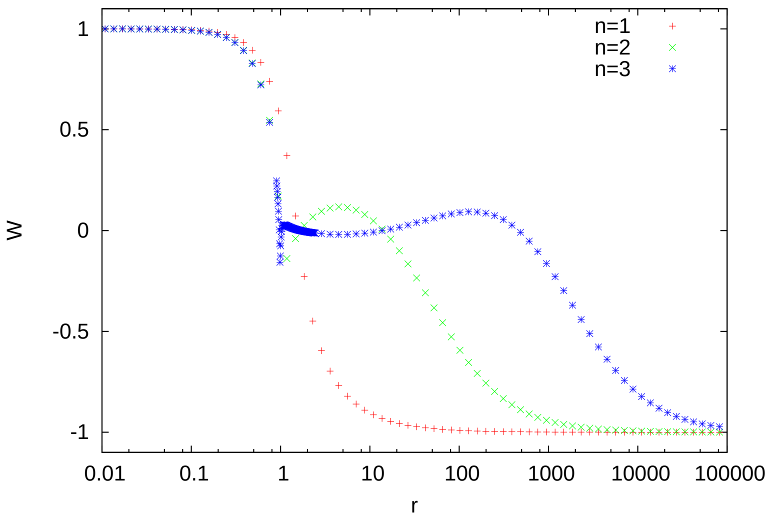

Figure 1: The first three Bartnik-McKinnon solutions generated in Maple. The $n = 1$ solution corresponds to $W_2 \approx -.453743610$ while the $n = 2$ and $n = 3$ solutions correspond to $W_2 \approx -.697070562$ and $W_2 \approx -.706199269$ respectively.

1.4 Boundary and Initial Conditions

From the above sections, it can be seen that the required boundary conditions are can be found by specifying $W(r,0)$ with $W(0,0)$ and $W(\infty,0) = \pm 1$. We may then compute $\Phi(r,0)$, $\Pi(r,0)$, $a(r,0)$ and $\alpha(r,0)$ from the constraint equations and requiring the initial data to be approximately ingoing.

Far from the origin, the metric must be flat and in the absence of matter we must approach the Schwarzschild solution. Therefore, we require $\lim_{r\to\infty}\alpha(r,t) = \frac{1}{a(r,t)}$. In this limit the Sommerfeld boundary conditions become:

$$\dot{\Pi} = -\Pi^{'} \tag{37}$$

$$\dot{\Phi} = -\Phi^{'} \tag{38}$$

while the expression for and approximately ingoing initial conditions are:

$$W(r,0) = f(r) \tag{39}$$

$$\Phi = f' \tag{40}$$

$$\Pi = \frac{a}{\alpha} f' \tag{41}$$

Note that the initial conditions for Π require knowledge of a and α while specification of a and α requires knowledge of Π. Rather than solving this system iteratively, we assume when specifying the initial conditions that $\frac{a}{\alpha} \approx 1$, find Π then calculate a and α. Although far from exact, the initial conditions are, in the first place, only approximate and this does give a good approximation of initially ingoing initial data in the weak field limit.

2 Methodology

2.1 Numerics

The program was written in RNPL [5] using Crank-Nicholson differencing with implicit Kreiss-Oliger-style dissipation. The Hamiltonian and momentum constraints were implemented separately using an RK3 solver with the Hamiltonian constraint integrated from the known boundary condition at the origin to $r = r_{max}$ and the Momentum constraint integrated backwards from $r = r_{max}$ by requiring $\alpha(r_{max}, t) = 1/a(r_{max}, t)$ (i.e. the geometry far from the sources should be Schwarzschild).

Black holes are detected through the use of the "black hole function",

$$Z(r,t) = \frac{2m(r,t)}{r} = \left(1 - a^{-2}\right) \tag{42}$$

which terminates the execution of the program when a user defined threshold (indicating imminent black hole formation $Z = 0.95$) is reached.

2.2 Initial Data

We consider the following initial data for W with Φ and Π subject to the initial conditions defined in the previous section:

$$W_0 = \frac{1 + \left(r_0^2 - r^2\right)/\delta^2}{\left(\left(1 + \left(r_0^2 - r^2\right)/\delta^2\right)^2 + 4r^2\right)^{1/2}} \tag{43}$$

where $r_0 = 30$ describes the location of the kink and δ is an adjustable parameter describing the width of the kink.

3 Results

Starting with very subcritical (dispersive) and very supercritical (quick black hole formation) solutions, a binary search in δ was performed to hone in on a value which resulted in the formation of the critical solution. Figure 2 shows the evolution of slightly supercritical and subcritical solutions compared to the static critical solution. Before dispersing or collapsing, the solutions approach the critical solution and remain there for a time after radiating excess Yang-Mills radiation. Videos of the evolution may be seen on my website [**?**].

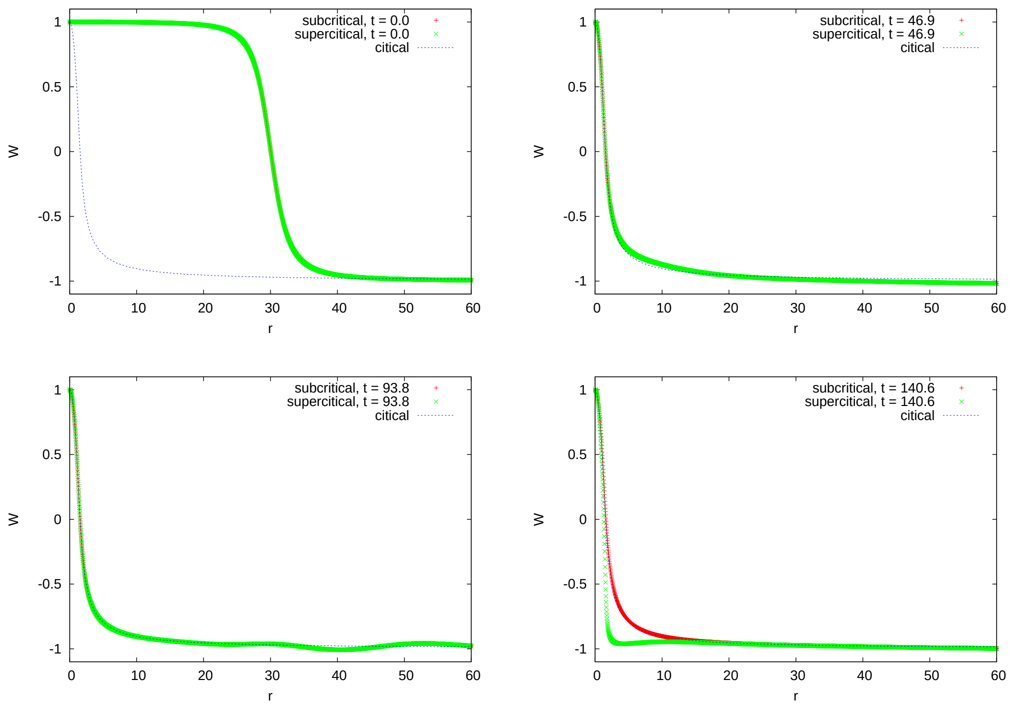

Figure 2: Evolution of a slightly supercritical and slightly subcritical initial configuration. After radiating excess Yang-Mills radiation, the two solutions approach the critical solution then diverge. Likely due to scattering off the approximate outgoing boundary conditions and small value of r_{max}, we were unable to find solutions which stuck close to the critical solution for long times.

References

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