GRAVITATION

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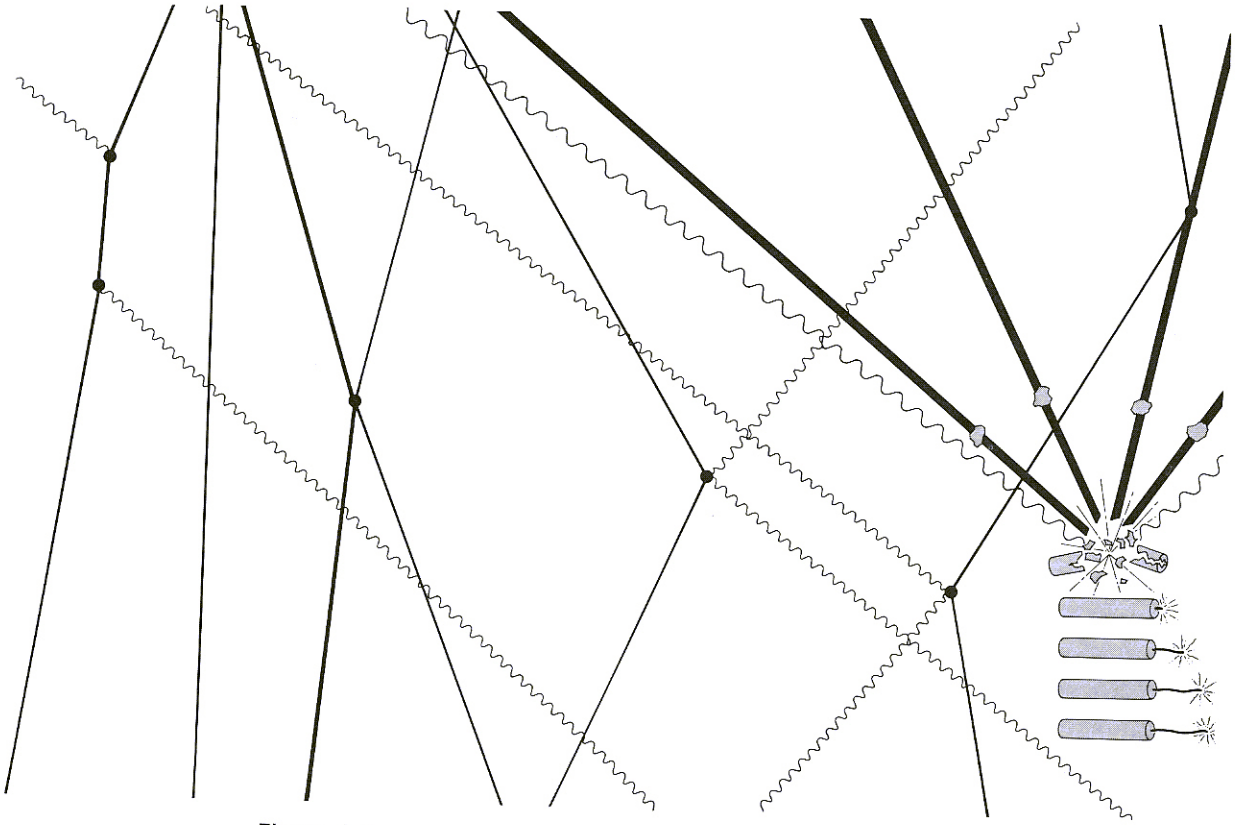

Figure 1.2.
The crossing of straws in a barn full of hay is a symbol for the world lines that fill up spacetime. By their crossings and bends, these world lines mark events with a uniqueness beyond all need of coordinate systems or coordinates. Typical events symbolized in the diagram, from left to right (black dots), are: absorption of a photon; reemission of a photon; collision between a particle and a particle; collision between a photon and a particle; another collision between a photon and a particle; explosion of a firecracker; and collision of a particle from outside with one of the fragments of that firecracker.

tack would record and reveal the shape of the surface. The precision could be made arbitrarily great by making the number of tacks arbitrarily large. It takes more daring to think of driving several score pitons into the towering iceberg. But with all the daring in the world, how is one to drive a nail into spacetime to mark a point? Happily, nature provides its own way to localize a point in spacetime, as Einstein was the first to emphasize. Characterize the point by what happens there! Give a point in spacetime the name "event." Where the event lies is defined as clearly and sharply as where two straws cross each other in a barn full of hay (Figure 1.2). To say that the event marks a collision of such and such a photon with such and such a particle is identification enough. The world lines of that photon and that particle are rooted in the past and stretch out into the future. They have a rich texture of connections with nearby world lines. These nearby world lines in turn are linked in a hundred ways with world lines more remote. How then does one tell the location of an event? Tell first what world lines participate in the event. Next follow each

Resolution: characterize events by what happens there

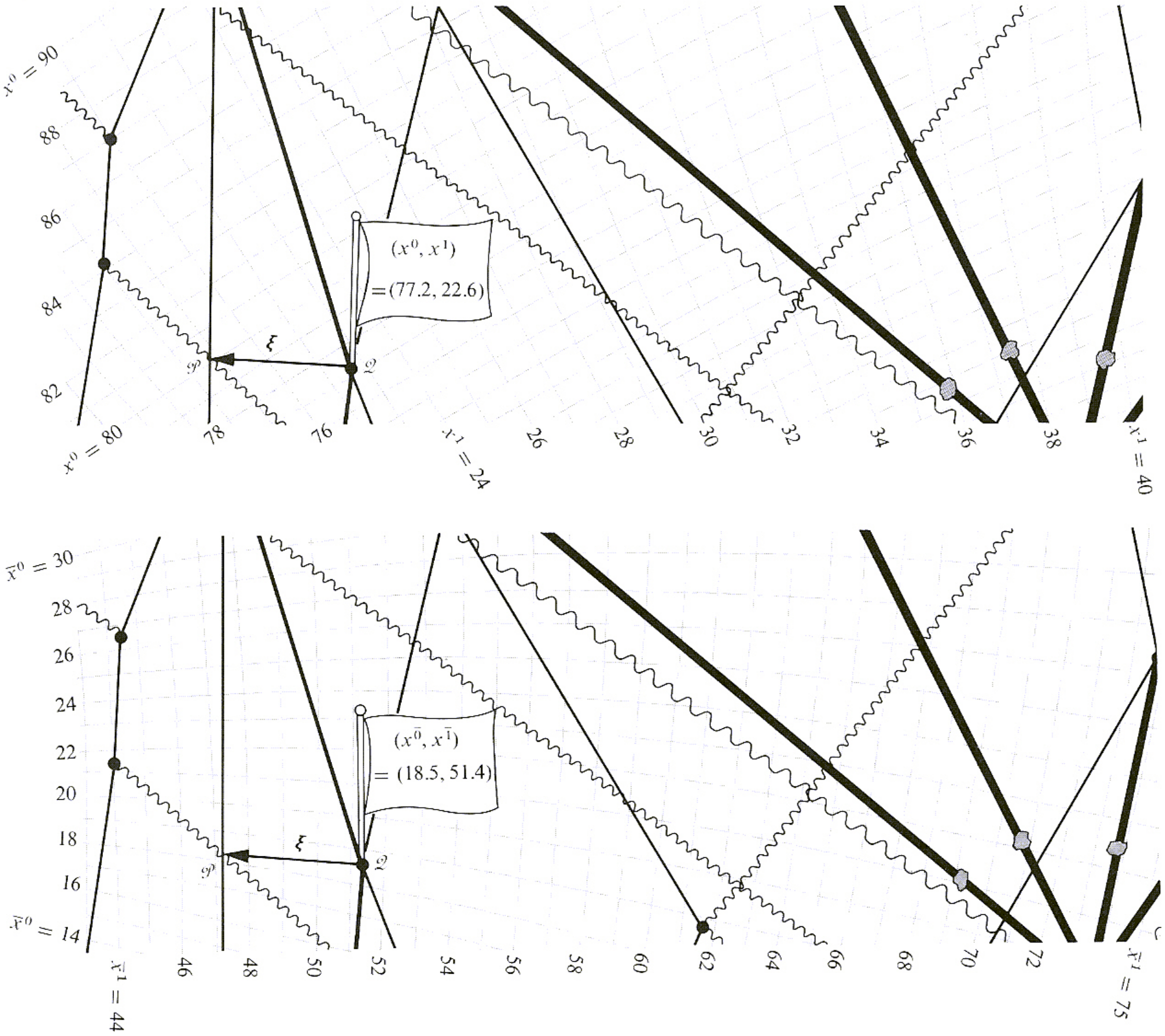

Figure 1.3.
Above: Assigning "telephone numbers" to events by way of a system of coordinates. To say that the coordinate system is "smooth" is to say that events which are almost in the same place have almost the same coordinates. Below: Putting the same set of events into equally good order by way of a different system of coordinates. Picked out specially here are two neighboring events: an event named "$\mathcal{Q}$" with coordinates $(x^0, x^1) = (77.2, 22.6)$ and $(x^{\bar{0}}, x^{\bar{1}}) = (18.5, 51.4)$; and an event named "$\mathcal{P}$" with coordinates $(x^0, x^1) = (79.9, 20.1)$ and $(x^{\bar{0}}, x^{\bar{1}}) = (18.4, 47.1)$. Events $\mathcal{Q}$ and $\mathcal{P}$ are connected by the separation "vector" $\boldsymbol{\xi}$. (Precise definition of a vector in a curved spacetime demands going to the mathematical limit in which the two points have an indefinitely small separation [N-fold reduction of the separation $\mathcal{P} - \mathcal{Q}$], and, in the resultant locally flat space, multiplying the separation up again by the factor N [lim $N \to \infty$; "tangent space"; "tangent vector"]. Forego here that proper way of stating matters, and forego complete accuracy; hence the quote around the word "vector".) In each coordinate system the separation vector $\boldsymbol{\xi}$ is characterized by "components" (differences in coordinate values between $\mathcal{P}$ and $\mathcal{Q}$):

$$(\xi^0, \xi^1) = (79.9 - 77.2,\ 20.1 - 22.6) = (2.7,\ -2.5),$$

$$(\xi^{\bar{0}}, \xi^{\bar{1}}) = (18.4 - 18.5,\ 47.1 - 51.4) = (-0.1,\ -4.3).$$

See Box 1.1 for further discussion of events, coordinates, and vectors.

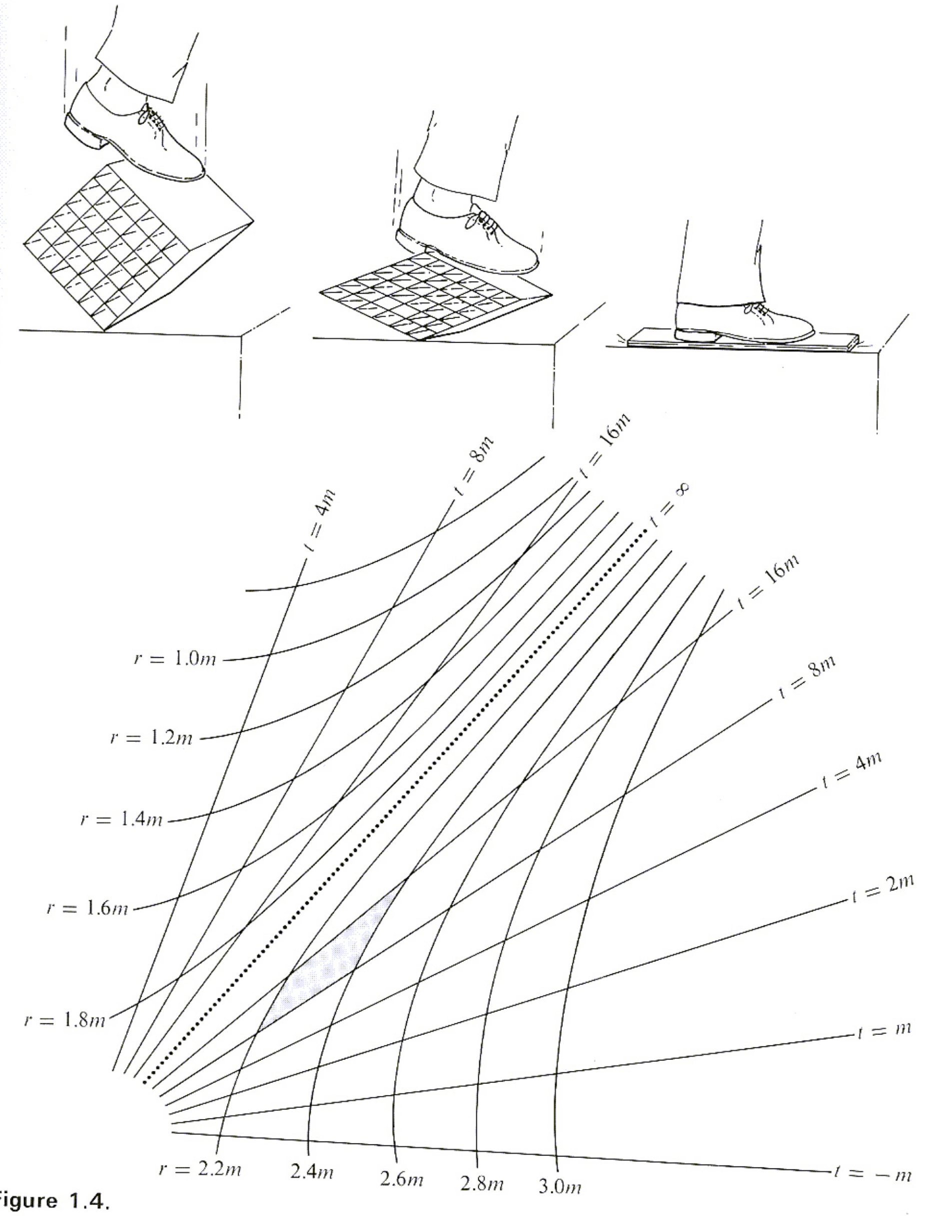

Figure 1.4.

How a mere coordinate singularity arises. Above: A coordinate system becomes *singular* when the "cells in the egg crate" are squashed to zero volume. Below: An example showing such a singularity in the Schwarzschild coordinates r, t often used to describe the geometry around a black hole (Chapter 31). For simplicity the angular coordinates θ, ϕ have been suppressed. The singularity shows itself in two ways. First, all the points along the dotted line, while quite distinct one from another, are designated by the same pair of (r, t) values; namely, $r = 2m$, $t = \infty$. The coordinates provide no way to distinguish these points. Second, the "cells in the egg crate," of which one is shown grey in the diagram, collapse to zero content at the dotted line. In summary, there is nothing strange about the geometry at the dotted line; all the singularity lies in the coordinate system ("poor system of telephone numbers"). No confusion should be permitted to arise from the accidental circumstance that the t coordinate attains an infinite value on the dotted line. No such infinity would occur if t were replaced by the new coordinate $\bar{t}$, defined by

$$(t/2m) = \tan(\bar{t}/2m).$$

When $t = \infty$, the new coordinate $\bar{t}$ is $\bar{t} = \pi m$. The r, $\bar{t}$ coordinates still provide no way to distinguish the points along the dotted line. They still give "cells in the egg crate" collapsed to zero content along the dotted line.

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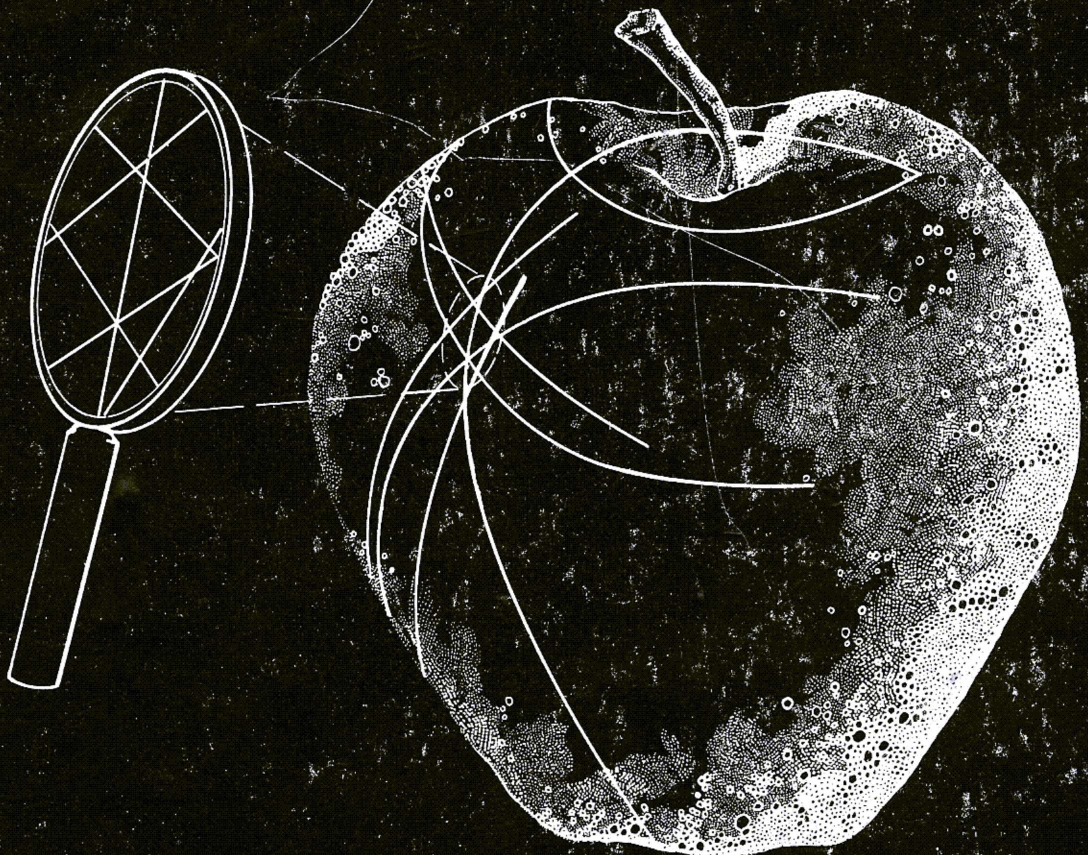


Box 1.3 LOCAL LORENTZ GEOMETRY AND LOCAL EUCLIDEAN GEOMETRY: WITH AND WITHOUT COORDINATES

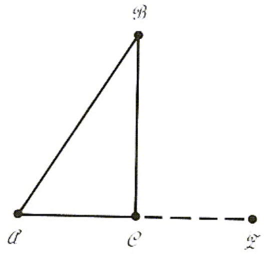
I. Local Euclidean Geometry

What does it mean to say that the geometry of a tiny thumbprint on the apple is Euclidean?

A. *Coordinate-free language* (Euclid):
Given a line $\mathcal{AC}$. Extend it by an equal distance $\mathcal{CZ}$. Let $\mathcal{B}$ be a point not on $\mathcal{AZ}$ but equidistant from $\mathcal{A}$ and $\mathcal{Z}$. Then

$$s_{\mathcal{AB}}^2 = s_{\mathcal{AC}}^2 + s_{\mathcal{BC}}^2.$$

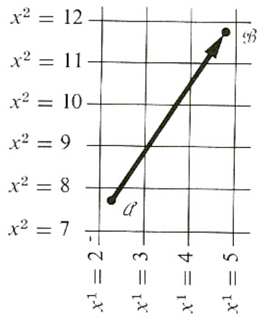
(Theorem of Pythagoras; also other theorems of Euclidean geometry.)



B. *Language of coordinates* (Descartes):
From any point $\mathcal{A}$ to any other point $\mathcal{B}$ there is a distance s given in suitable (Euclidean) coordinates by

$$s_{\mathcal{AB}}^2 = [x^1(\mathcal{B}) - x^1(\mathcal{A})]^2 + [x^2(\mathcal{B}) - x^2(\mathcal{A})]^2.$$

If one succeeds in finding any coordinate system where this is true for all points $\mathcal{A}$ and $\mathcal{B}$ in the thumbprint, then one is guaranteed that (i) this coordinate system is locally Euclidean, and (ii) the geometry of the apple's surface is locally Euclidean.

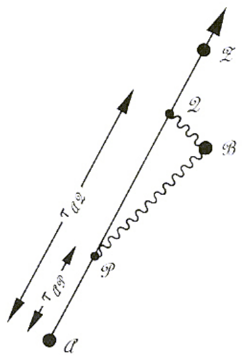


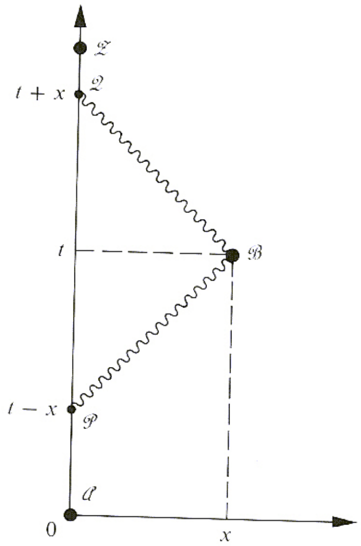
II. Local Lorentz Geometry

What does it mean to say that the geometry of a sufficiently limited region of spacetime in the real physical world is Lorentzian?

A. *Coordinate-free language* (Robb 1936):
Let $\mathcal{AZ}$ be the world line of a free particle. Let $\mathcal{B}$ be an event not on this world line. Let a light ray from $\mathcal{B}$ strike $\mathcal{AZ}$ at the event $\mathcal{Q}$. Let a light ray take off from such an earlier event $\mathcal{P}$ along $\mathcal{AZ}$ that it reaches $\mathcal{B}$. Then the proper distance $s_{\mathcal{AB}}$ (spacelike separation) or proper time $\tau_{\mathcal{AB}}$ (timelike separation) is given by

$$s_{\mathcal{AB}}^2 \equiv -\tau_{\mathcal{AB}}^2 = -\tau_{\mathcal{AQ}}\tau_{\mathcal{AP}}.$$



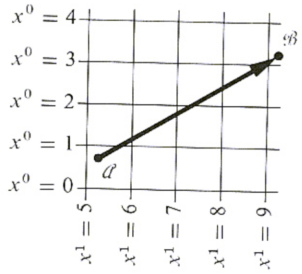


Proof of above criterion for local Lorentz geometry, using coordinate methods in the local Lorentz frame where particle remains at rest:

$$\tau_{\mathscr{A}\mathscr{B}}{}^2 = t^2 - x^2 = (t-x)(t+x) = \tau_{\mathscr{A}\mathscr{P}}\tau_{\mathscr{A}\mathscr{Q}}.$$

B. *Language of coordinates* (Lorentz, Poincaré, Minkowski, Einstein):
From any event $\mathscr{A}$ to any other nearby event $\mathscr{B}$, there is a proper distance $s_{\mathscr{A}\mathscr{B}}$ or proper time $\tau_{\mathscr{A}\mathscr{B}}$ given in suitable (local Lorentz) coordinates by

$$\begin{aligned} s_{\mathscr{A}\mathscr{B}}{}^2 = -\tau_{\mathscr{A}\mathscr{B}}{}^2 = &-[x^0(\mathscr{B}) - x^0(\mathscr{A})]^2 \\ &+ [x^1(\mathscr{B}) - x^1(\mathscr{A})]^2 \\ &+ [x^2(\mathscr{B}) - x^2(\mathscr{A})]^2 \\ &+ [x^3(\mathscr{B}) - x^3(\mathscr{A})]^2. \end{aligned}$$



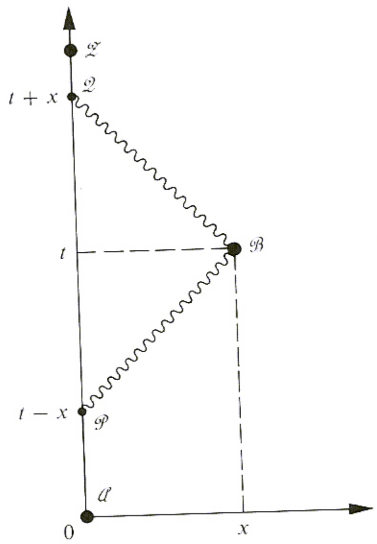
If one succeeds in finding any coordinate system where this is locally true for all neighboring events $\mathscr{A}$ and $\mathscr{B}$, then one is guaranteed that (i) this coordinate system is locally Lorentzian, and (ii) the geometry of spacetime is locally Lorentzian.

III. Statements of Fact

The geometry of an apple's surface is locally Euclidean everywhere. The geometry of spacetime is locally Lorentzian everywhere.

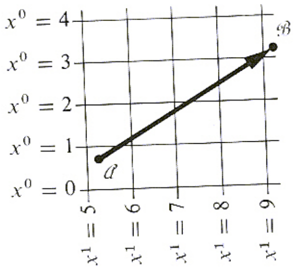
Proof of above criterion for local Lorentz geometry, using coordinate methods in the local Lorentz frame where particle remains at rest:

$$\tau_{\mathcal{AB}}{}^2 = t^2 - x^2 = (t - x)(t + x) = \tau_{\mathcal{AP}}\tau_{\mathcal{AQ}}.$$



B. *Language of coordinates* (Lorentz, Poincaré, Minkowski, Einstein):
From any event $\mathcal{A}$ to any other nearby event $\mathcal{B}$, there is a proper distance $s_{\mathcal{AB}}$ or proper time $\tau_{\mathcal{AB}}$ given in suitable (local Lorentz) coordinates by

$$s_{\mathcal{AB}}{}^2 = -\tau_{\mathcal{AB}}{}^2 = -[x^0(\mathcal{B}) - x^0(\mathcal{A})]^2 + [x^1(\mathcal{B}) - x^1(\mathcal{A})]^2 + [x^2(\mathcal{B}) - x^2(\mathcal{A})]^2 + [x^3(\mathcal{B}) - x^3(\mathcal{A})]^2.$$



If one succeeds in finding any coordinate system where this is locally true for all neighboring events $\mathcal{A}$ and $\mathcal{B}$, then one is guaranteed that (i) this coordinate system is locally Lorentzian, and (ii) the geometry of spacetime is locally Lorentzian.

III. Statements of Fact

The geometry of an apple's surface is locally Euclidean everywhere. The geometry of spacetime is locally Lorentzian everywhere.

Box 1.3 (*continued*)

IV. Local Geometry in the Language of Modern Mathematics

A. *The metric for any manifold:*
At each point on the apple, at each event of spacetime, indeed, at each point of any "Riemannian manifold," there exists a geometrical object called the *metric tensor* $\boldsymbol{g}$. It is a machine with two input slots for the insertion of two vectors:

$$\overset{\text{slot 1}}{\downarrow}\quad \overset{\text{slot 2}}{\downarrow}$$
$$\boldsymbol{g}(\quad ,\quad).$$

If one inserts the same vector $\boldsymbol{u}$ into both slots, one gets out the square of the length of $\boldsymbol{u}$:

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{u}^2.$$

If one inserts two different vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$ (it matters not in which order!), one gets out a number called the "scalar product of $\boldsymbol{u}$ on $\boldsymbol{v}$" and denoted $\boldsymbol{u} \cdot \boldsymbol{v}$:

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}.$$

The metric is a linear machine:

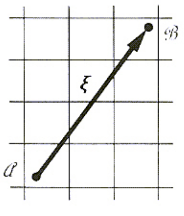
$$\boldsymbol{g}(2\boldsymbol{u} + 3\boldsymbol{w}, \boldsymbol{v}) = 2\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) + 3\boldsymbol{g}(\boldsymbol{w}, \boldsymbol{v}),$$
$$\boldsymbol{g}(\boldsymbol{u}, a\boldsymbol{v} + b\boldsymbol{w}) = a\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) + b\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{w}).$$

Consequently, in a given (arbitrary) coordinate system, its operation on two vectors can be written in terms of their components as a bilinear expression:

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\alpha\beta}u^\alpha v^\beta \quad \text{(implied summation on } \alpha, \beta)$$
$$= g_{11}u^1v^1 + g_{12}u^1v^2 + g_{21}u^2v^1 + \cdots.$$

The quantities $g_{\alpha\beta} = g_{\beta\alpha}$ (α and β running from 0 to 3 in spacetime, from 1 to 2 on the apple) are called the "components of $\boldsymbol{g}$ in the given coordinate system."

B. *Components of the metric in local Lorentz and local Euclidean frames:*
To connect the metric with our previous descriptions of the local geometry, introduce local Euclidean coordinates (on apple) or local Lorentz coordinates (in spacetime).



Let $\boldsymbol{\xi}$ be the separation vector reaching from $\mathcal{A}$ to $\mathcal{B}$. Its components in the local Euclidean (Lorentz) coordinates are

$$\xi^\alpha = x^\alpha(\mathcal{B}) - x^\alpha(\mathcal{A})$$

(cf. Box 1.1). Then the squared length of $\boldsymbol{u}_{\mathcal{AB}}$, which is the same as the squared distance from $\mathcal{A}$ to $\mathcal{B}$, must be (cf. I.B. and II.B. above)

$$\begin{aligned}\boldsymbol{\xi} \cdot \boldsymbol{\xi} &= \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) = g_{\alpha\beta}\xi^\alpha\xi^\beta \\ &= s_{\mathcal{AB}}{}^2 = (\xi^1)^2 + (\xi^2)^2 \text{ on apple} \\ &= -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \text{ in spacetime.}\end{aligned}$$

Consequently, the components of the metric are

$$g_{11} = g_{22} = 1,\ g_{12} = g_{21} = 0; \quad \text{i.e., } g_{\alpha\beta} = \delta_{\alpha\beta} \text{ on apple, in local Euclidean coordinates;}$$
$$g_{00} = -1,\ g_{0k} = 0,\ g_{jk} = \delta_{jk} \text{ in spacetime, in local Lorentz coordinates.}$$

These special components of the metric in local Lorentz coordinates are written here and hereafter as $g_{\hat\alpha\hat\beta}$ or $\eta_{\alpha\beta}$, by analogy with the Kronecker delta $\delta_{\alpha\beta}$. In matrix notation:

$$\|g_{\hat\alpha\hat\beta}\| = \|\eta_{\alpha\beta}\| = \begin{array}{c|cccc} \alpha \backslash \beta & 0 & 1 & 2 & 3 \\ \hline 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

to empirical test in the appropriate, very special coordinate systems: Euclidean coordinates in Euclidean geometry; the natural generalization of Euclidean coordinates (local Lorentz coordinates; local inertial frame) in the local Lorentz geometry of physics. However, the theorems rise above all coordinate systems in their content. They refer to intervals or distances. Those distances no more call on coordinates for their definition in our day than they did in the time of Euclid. Points in the great pile of hay that is spacetime; and distances between these points: that is geometry! State them in the coordinate-free language or in the language of coordinates: they are the same (Box 1.3).

§1.5. TIME

Time is defined so that motion looks simple.

Time is awake when all things sleep.
Time stands straight when all things fall.
Time shuts in all and will not be shut.
Is, was, and shall be are Time's children.
O Reasoning, be witness, be stable.

VYASA, the *Mahabarata* (*ca.* A.D. 400)

Relative to a local Lorentz frame, a free particle "moves in a straight line with uniform velocity." What "straight" means is clear enough in the model inertial reference frame illustrated in Figure 1.7. But where does the "uniform velocity" come in? Or where does "velocity" show itself? There is not even one clock in the drawing!

A more fully developed model of a Lorentz reference frame will have not only holes, as in Fig. 1.7, but also clock-activated shutters over each hole. The projectile can reach its target only if it (1) travels through the correct region in space and (2) gets through that hole in the correct interval of time ("window in time"). How then is time defined? Time is defined so that motion looks simple!

The time coordinate of a local Lorentz frame is so defined that motion looks simple

No standard of time is more widely used than the day, the time from one high noon to the next. Take that as standard, however, and one will find every good clock or watch clashing with it, for a simple reason. The Earth spins on its axis and also revolves in orbit about the sun. The motion of the sun across the sky arises from neither effect alone, but from the two in combination, different in magnitude though they are. The fast angular velocity of the Earth on its axis (roughly 366.25 complete turns per year) is wonderfully uniform. Not so the apparent angular velocity of the sun about the center of the Earth (one turn per year). It is greater than average by 2 per cent when the Earth in its orbit (eccentricity 0.017) has come 1 per cent closer than average to the sun (Kepler's law) and lower by 2 per cent when the Earth is 1 per cent further than average from the sun. In the first case, the momentary rate of rotation of the sun across the sky, expressed in turns per year, is approximately

$$366.25 - (1 + 0.02);$$

CHAPTER **2**

FOUNDATIONS OF SPECIAL RELATIVITY

In geometric and physical applications, it always turns out that a quantity is characterized not only by its tensor order, but also by symmetry.

HERMAN WEYL (1925)

Undoubtedly the most striking development of geometry during the last 2,000 years is the continual expansion of the concept "geometric object." This concept began by comprising only the few curves and surfaces of Greek synthetic geometry; it was stretched, during the Renaissance, to cover the whole domain of those objects defined by analytic geometry; more recently, it has been extended to cover the boundless universe treated by point-set theory.

KARL MENGER, IN SCHILPP (1949), P. 466.

§2.1. OVERVIEW

Curvature in geometry manifests itself as gravitation. Gravitation works on the separation of nearby particle world lines. In turn, particles and other sources of mass-energy cause curvature in the geometry. How does one break into this closed loop of the action of geometry on matter and the reaction of matter on geometry? One can begin no better than by analyzing the motion of particles and the dynamics of fields in a region of spacetime so limited that it can be regarded as flat. (See "Test for Flatness," Box 1.5).

Background assumed of reader

Chapters 2–6 develop this flat-spacetime viewpoint (special relativity). The reader, it is assumed, is already somewhat familiar with special relativity:* 4-vectors in general; the energy-momentum 4-vector; elementary Lorentz transformations; the Lorentz law for the force on a charged particle; at least one look at one equation

*For example, see Goldstein (1959), Leighton (1959), Jackson (1962), or, for the physical perspective presented geometrically, Taylor and Wheeler (1966).

in one book that refers to the electromagnetic field tensor $F_{\mu\nu}$; and the qualitative features of spacetime diagrams, including such points as (1) future and past light cones, (2) causal relationships ("past of," "future of," "neutral," or "in a spacelike relationship to"), (3) Lorentz contraction, (4) time dilation, (5) absence of a universal concept of simultaneity, and (6) the fact that the $\bar{t}$ and $\bar{z}$ axes in Box 2.4 are orthogonal even though they do not look so. If the reader finds anything new in these chapters, it will be: (i) a new viewpoint on special relativity, one emphasizing coordinate-free concepts and notation that generalize readily to curved spacetime ("geometric objects," tensors viewed as machines—treated in Chapters 2–4); or (ii) unfamiliar topics in special relativity, topics crucial to the later exposition of gravitation theory ("stress-energy tensor and conservation laws," Chapter 5; "accelerated observers," Chapter 6).

§2.2. GEOMETRIC OBJECTS

Every physical quantity can be described by a geometric object

Everything that goes on in spacetime has its geometric description, and almost every one of these descriptions lends itself to ready generalization from flat spacetime to curved spacetime. The greatest of the differences between one geometric object and another is its scope: the individual object (vector) for the momentum of a certain particle at a certain phase in its history, as contrasted to the extended geometric object that describes an electromagnetic field defined throughout space and time ("antisymmetric second-rank tensor field" or, more briefly, "field of 2-forms"). The idea that every physical quantity must be describable by a geometric object, and that the laws of physics must all be expressible as geometric relationships between these geometric objects, had its intellectual beginnings in the Erlanger program of Felix Klein (1872), came closer to physics in Einstein's "principle of general covariance" and in the writings of Hermann Weyl (1925), seems to have first been formulated clearly by Veblen and Whitehead (1932), and today pervades relativity theory, both special and general.

All laws of physics can be expressed geometrically

A. Nijenhuis (1952) and S.-S. Chern (1960, 1966, 1971) have expounded the mathematical theory of geometric objects. But to understand or do research in geometrodynamics, one need not master this elegant and beautiful subject. One need only know that geometric objects in spacetime are entities that exist independently of coordinate systems or reference frames. A point in spacetime ("*event*") is a geometric object. The arrow linking two neighboring events ("*vector*") is a geometric object in flat spacetime, and its generalization, the "*tangent vector,*" is a geometric object even when spacetime is curved. The "*metric*" (machine for producing the squared length of any vector; see Box 1.3) is a geometric object. No coordinates are needed to define any of these concepts.

The next few sections will introduce several geometric objects, and show the roles they play as representatives of physical quantities in flat spacetime.

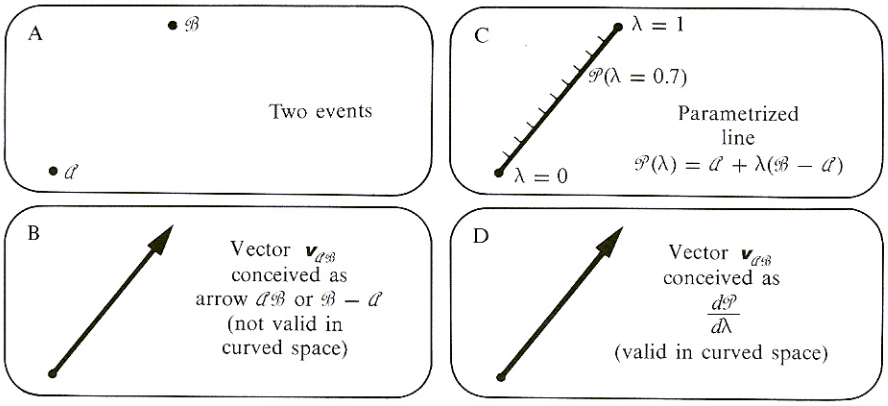


Figure 2.1.
From vector as connector of two points to vector as derivative ("tangent vector"; a local rather than a bilocal concept).

§2.3. VECTORS

Begin with the simplest idea of a vector (Figure 2.1B): an arrow extending from one spacetime event $\mathcal{A}$ ("tail") to another event $\mathcal{B}$ ("tip"). Write this vector as

Ways of defining vector: As arrow

$$\boldsymbol{v}_{\mathcal{AB}} = \mathcal{B} - \mathcal{A} \text{ (or } \mathcal{AB}).$$

For many purposes (including later generalization to curved spacetime) other completely equivalent ways to think of this vector are more convenient. Represent the arrow by the parametrized straight line $\mathcal{P}(\lambda) = \mathcal{A} + \lambda(\mathcal{B} - \mathcal{A})$, with $\lambda = 0$ the tail of the arrow, and $\lambda = 1$ its tip. Form the derivative of this simple linear expression for $\mathcal{P}(\lambda)$:

As parametrized straight line

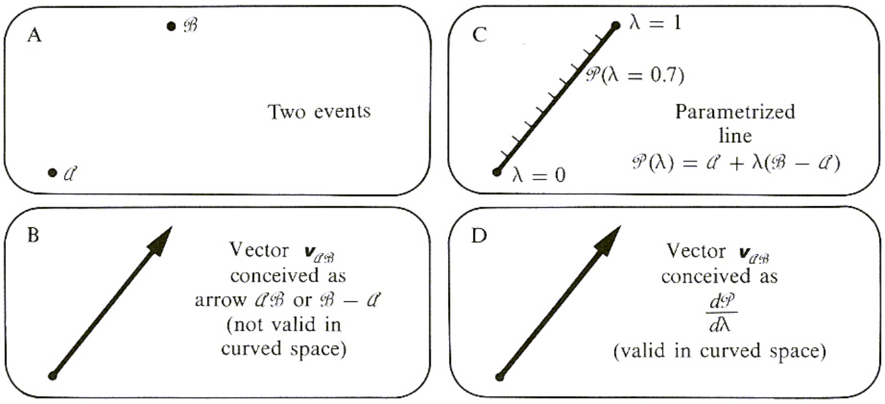
$$(d/d\lambda)[\mathcal{A} + \lambda(\mathcal{B} - \mathcal{A})] = \mathcal{B} - \mathcal{A} = \mathcal{P}(1) - \mathcal{P}(0) \equiv \text{(tip)} - \text{(tail)} \equiv \boldsymbol{v}_{\mathcal{AB}}.$$

This result allows one to replace the idea of a vector as a 2-point object ("bilocal") by the concept of a vector as a 1-point object ("tangent vector"; local):

$$\boldsymbol{v}_{\mathcal{AB}} = (d\mathcal{P}/d\lambda)_{\lambda = 0}. \quad (2.1)$$

As derivative of point along curve

Example: if $\mathcal{P}(\tau)$ is the straight world line of a free particle, parametrized by its proper time, then the displacement that occurs in a proper time interval of one second gives an arrow $\boldsymbol{u} = \mathcal{P}(1) - \mathcal{P}(0)$. This arrow is easily drawn on a spacetime diagram. It accurately shows the 4-velocity of the particle. However, the derivative formula $\boldsymbol{u} = d\mathcal{P}/d\tau$ for computing the same displacement (1) is more suggestive of the velocity concept and (2) lends itself to the case of accelerated motion. Thus, given a world line $\mathcal{P}(\tau)$ that is not straight, as in Figure 2.2, one must first form $d\mathcal{P}/d\tau$, and only thereafter draw the straight line $\mathcal{P}(0) + \lambda(d\mathcal{P}/d\tau)_0$ of the arrow $\boldsymbol{u} = d\mathcal{P}/d\tau$ to display the 4-velocity $\boldsymbol{u}$.



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As parametrized straight line

$$(d/d\lambda)[\mathscr{A} + \lambda(\mathscr{B} - \mathscr{A})] = \mathscr{B} - \mathscr{A} = \mathscr{P}(1) - \mathscr{P}(0) \equiv \text{(tip)} - \text{(tail)} \equiv \boldsymbol{v}_{\mathscr{A}\mathscr{B}}.$$

This result allows one to replace the idea of a vector as a 2-point object ("bilocal") by the concept of a vector as a 1-point object ("tangent vector"; local):

$$\boldsymbol{v}_{\mathscr{A}\mathscr{B}} = (d\mathscr{P}/d\lambda)_{\lambda=0}. \tag{2.1}$$

As derivative of point along curve

Example: if $\mathscr{P}(\tau)$ is the straight world line of a free particle, parametrized by its proper time, then the displacement that occurs in a proper time interval of one second gives an arrow $\boldsymbol{u} = \mathscr{P}(1) - \mathscr{P}(0)$. This arrow is easily drawn on a spacetime diagram. It accurately shows the 4-velocity of the particle. However, the derivative formula $\boldsymbol{u} = d\mathscr{P}/d\tau$ for computing the same displacement (1) is more suggestive of the velocity concept and (2) lends itself to the case of accelerated motion. Thus, given a world line $\mathscr{P}(\tau)$ that is not straight, as in Figure 2.2, one must first form $d\mathscr{P}/d\tau$, and only thereafter draw the straight line $\mathscr{P}(0) + \lambda(d\mathscr{P}/d\tau)_0$ of the arrow $\boldsymbol{u} = d\mathscr{P}/d\tau$ to display the 4-velocity $\boldsymbol{u}$.

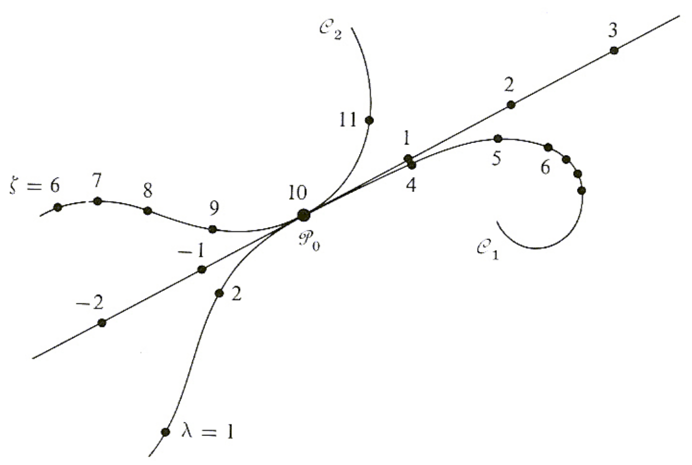

Figure 2.2.
Same tangent vector derived from two very different curves. That parametrized straight line is also drawn which best fits the two curves at $\mathscr{P}_0$. The tangent vector reaches from 0 to 1 on this straight line.

Components of a vector

The reader may be unfamiliar with this viewpoint. More familiar may be the components of the 4-velocity in a specific Lorentz reference frame:

$$u^0 = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}}, \qquad u^j = \frac{dx^j}{d\tau} = \frac{v^j}{\sqrt{1-v^2}}, \tag{2.2}$$

where

$$v^j = dx^j/dt = \text{components of “ordinary velocity,”}$$
$$v^2 = (v^x)^2 + (v^y)^2 + (v^z)^2.$$

Even the components (2.2) of 4-velocity may seem slightly unfamiliar if the reader is accustomed to having the fourth component of a vector be multiplied by a factor $i = \sqrt{-1}$. If so, he must adjust himself to new notation. (See “Farewell to ‘*ict*,’” Box 2.1.)

More fundamental than the components of a vector is the vector itself. It is a geometric object with a meaning independent of all coordinates. Thus a particle has a world line $\mathscr{P}(\tau)$, and a 4-velocity $\boldsymbol{u} = d\mathscr{P}/d\tau$, that have nothing to do with any coordinates. Coordinates enter the picture when analysis on a computer is required (rejects vectors; accepts numbers). For this purpose one adopts a Lorentz frame with orthonormal basis vectors (Figure 2.3) $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. Relative to the origin $\mathscr{O}$ of this frame, the world line has a coordinate description

Basis vectors

$$\mathscr{P}(\tau) - \mathscr{O} = x^0(\tau)\boldsymbol{e}_0 + x^1(\tau)\boldsymbol{e}_1 + x^2(\tau)\boldsymbol{e}_2 + x^3(\tau)\boldsymbol{e}_3 = x^\mu(\tau)\boldsymbol{e}_\mu.$$

Expressed relative to the same Lorentz frame, the 4-velocity of the particle is

$$\boldsymbol{u} = d\mathscr{P}/d\tau = (dx^\mu/d\tau)\boldsymbol{e}_\mu = u^0\boldsymbol{e}_0 + u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 + u^3\boldsymbol{e}_3. \tag{2.3}$$

Box 2.1 FAREWELL TO *"ict"*

One sometime participant in special relativity will have to be put to the sword: "$x^4 = ict$." This imaginary coordinate was invented to make the geometry of spacetime look formally as little different as possible from the geometry of Euclidean space; to make a Lorentz transformation look on paper like a rotation; and to spare one the distinction that one otherwise is forced to make between quantities with upper indices (such as the components p^μ of the energy-momentum vector) and quantities with lower indices (such as the components p_μ of the energy-momentum 1-form). However, it is no kindness to be spared this latter distinction. Without it, one cannot know whether a vector (§2.3) is meant or the very different geometric object that is a 1-form (§2.5). Moreover, there is a significant difference between an angle on which everything depends periodically (a rotation) and a parameter the increase of which gives rise to ever-growing momentum differences (the "velocity parameter" of a Lorentz transformation; Box 2.4). If the imaginary time-coordinate hides from view the character of the geometric object being dealt with and the nature of the parameter in a transformation, it also does something even more serious: it hides the completely different metric structure (§2.4) of $+++$ geometry and $-+++$ geometry. In Euclidean geometry, when the distance between two points is zero, the two points must be the same point. In Lorentz-Minkowski geometry, when the interval between two events is zero, one event may be on Earth and the other on a supernova in the galaxy M31, but their separation must be a null ray (piece of a light cone). The backward-pointing light cone at a given event contains all the events by which that event can be influenced. The forward-pointing light cone contains all events that it can influence. The multitude of double light cones taking off from all the events of spacetime forms an interlocking causal structure. This structure makes the machinery of the physical world function as it does (further comments on this structure in Wheeler and Feynman 1945 and 1949 and in Zeeman 1964). If in a region where spacetime is flat, one can hide this structure from view by writing

$$(\Delta s)^2 = (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 + (\Delta x^4)^2,$$

with $x^4 = ict$, no one has discovered a way to make an imaginary coordinate work in the general curved spacetime manifold. If "$x^4 = ict$" cannot be used there, it will not be used here. In this chapter and hereafter, as throughout the literature of general relativity, a real time coordinate is used, $x^0 = t = ct_{\text{conv}}$ (superscript 0 rather than 4 to avoid any possibility of confusion with the imaginary time coordinate).

The components w^α of any other vector $\boldsymbol{w}$ in this frame are similarly defined as the coefficients in such an expansion,

Expansion of vector in terms of basis

$$\boldsymbol{w} = w^\alpha \boldsymbol{e}_\alpha. \tag{2.4}$$

Notice: the subscript α on $\boldsymbol{e}_\alpha$ tells which vector, not which component!

§2.4. THE METRIC TENSOR

The metric tensor, one recalls from part IV of Box 1.3, is a machine for calculating the squared length of a single vector, or the scalar product of two different vectors.

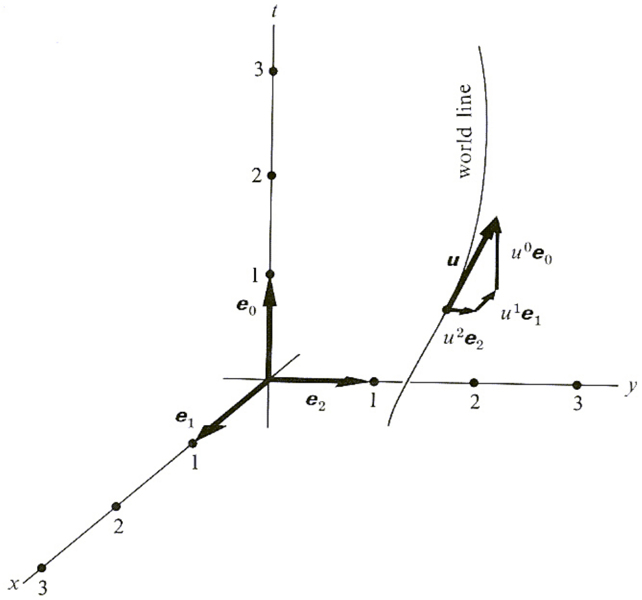

Figure 2.3.
The 4-velocity of a particle in flat spacetime. The 4-velocity $\boldsymbol{u}$ is the unit vector (arrow) tangent to the particle's world line—one tangent vector for each event on the world line. In a specific Lorentz coordinate system, there are basis vectors of unit length, which point along the four coordinate axes: $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The 4-velocity, like any vector, can be expressed as a sum of components along the basis vectors:

$$\boldsymbol{u} = u^0\mathbf{e}_0 + u^1\mathbf{e}_1 + u^2\mathbf{e}_2 + u^3\mathbf{e}_3 = u^\alpha\mathbf{e}_\alpha.$$

Metric defined as machine for computing scalar products of vectors

More precisely, the metric tensor $\boldsymbol{g}$ is a machine with two slots for inserting vectors

$$\begin{matrix} & \text{slot 1} & \text{slot 2} & \\ & \downarrow & \downarrow & \\ \boldsymbol{g}(& \quad, & &). \end{matrix} \tag{2.5}$$

Upon insertion, the machine spews out a real number:

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \text{“scalar product of } \boldsymbol{u} \text{ and } \boldsymbol{v}\text{,” also denoted } \boldsymbol{u}\cdot\boldsymbol{v}. \tag{2.6}$$
$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{u}) = \text{“squared length of } \boldsymbol{u}\text{,” also denoted } \boldsymbol{u}^2.$$

Moreover, this number is independent of the order in which the vectors are inserted ("symmetry of metric tensor"),

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u}); \tag{2.7}$$

and it is linear in the vectors inserted

$$\boldsymbol{g}(a\boldsymbol{u} + b\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{g}(\boldsymbol{w}, a\boldsymbol{u} + b\boldsymbol{v}) = a\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{w}) + b\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{w}). \tag{2.8}$$

Because the metric "machine" is linear, one can calculate its output, for any input,