THE 3+1 (ADM / HAMILTONIAN) FORM of GR

REFERENCES

(1) MTW CHAPTER 21

(2) YORK, "KINEMATICS & DYNAMICS of GR" IN "SOURCES of GRAV. RADIATION" (L. SMARR ed)

(3) WALD, APP. E.2 ; CHAPTER 10

(4) ARNOWITT, DESER & MISNER (1962) "THE DYNAMICS of GR" IN "GRAVITATION: AN INTRODUCTION TO CURRENT RESEARCH" (L. WITTEN ed)

- ADM MOTIVATION WAS, AGAIN, PREPARATION FOR QUANTIZATION of GR; FORMALISM TURNED OUT TO BE EXCELLENT BASIS FOR COMPUTATIONAL (NUMERICAL) ASSAULT ON EINSTEIN EQUATIONS

APPROACHES

(1) "COORDINATE-FUL" (MTW): INTUITIVE, CONNECTS MORE DIRECTLY TO FORM of E.O.M USED IN PRACTICE

(2) "COORDINATE-FREE" (YORK / WALD): PREFERABLE FOR DERIVATION of E.O.M.

KEY POINT: MUST INTRODUCE A COORDINATE SYSTEM TO NUMERICALLY GENERATE A SPACE-TIME; I.E. CAN'T STAY COORDINATE FREE FOREVER

- WILL ~~START~~ WITH COORDINATE BASED APPROACH TO INTRODUCE CONCEPTS, THEN WILL GO OVER TO COORDINATE-FREE APPROACH TO DERIVE 3+1 EQN'S

ULTIMATE GOAL: REFORMULATE

$$G_{ab} = 8\pi T_{ab}$$

AS SYSTEM of FIRST-ORDER (IN TIME) PDE'S FOR THE GRAV. FIELD VBL^{S}, WHICH CAN THEN BE SOLVED AS AN "INITIAL-VALUE" or "CAUCHY" PROBLEM

SHIFT IN PERSPECTIVE: UP TO NOW $G_{ab} = 8\pi T_{ab}$ DESCRIBED THE LINKAGE OF THE GEOMETRY of SPACETIME (4-D) TO THE DISTRIBUTION of MATTER-STRESS-ENERGY IN S.T.
NEW VIEW: GEOMETRY of S.T IS "TIME-HISTORY" (EVOLUTION) of GEOMETRY of A SPACELIKE HYPERSURFACE (3-D) $\Rightarrow$ GEOMETRODYNAMICS; "VIEW" NOT UNIQUE SINCE THERE ARE ∞'LY MANY WAYS of "SLICING UP" GIVEN S.T. INTO A FAMILY OF S.L. HYPERSURFACES; NONE PREFERRED (I.E. PHYSICALLY MORE RELEVANT) IN GENERAL

SPLITTING SPACETIME INTO SPACE-PLUS-TIME (THE 3+1 SPLIT)

- SPACETIME IS 4-DIMENSIONAL MANIFOLD M, WITH LORENTZIAN SIGNATURE METRIC g_{ab} $(-+++)$
- INTRODUCE COORDINATES $\{x^{\mu}\} = \{t, x^{i}\}$ (MAY NOT COVER ENTIRE S.T., BUT WILL GENERALLY PRETEND THEY DO)

NOTATION / CONVENTIONS:

GREEK INDICES (μ, ν etc.) 0, 1, 2, 3

"INTEGER" LATIN " (i, j, k, l, m, n) 1, 2, 3 (SPATIAL)

WILL ADOPT USUAL EINSTEIN SUMMATION CONVENTION FOR BOTH TYPES!

- DEMAND THAT t = CONST SURFACES (HYPERSURFACES), $\Sigma(t)$, ARE SPACELIKE; I.E. IF DISTINCT EVENTS P, P'

HAVE COORDS (t, x^i), $(t, x^{i'})$ THEN $ds^2(P, P') > 0$

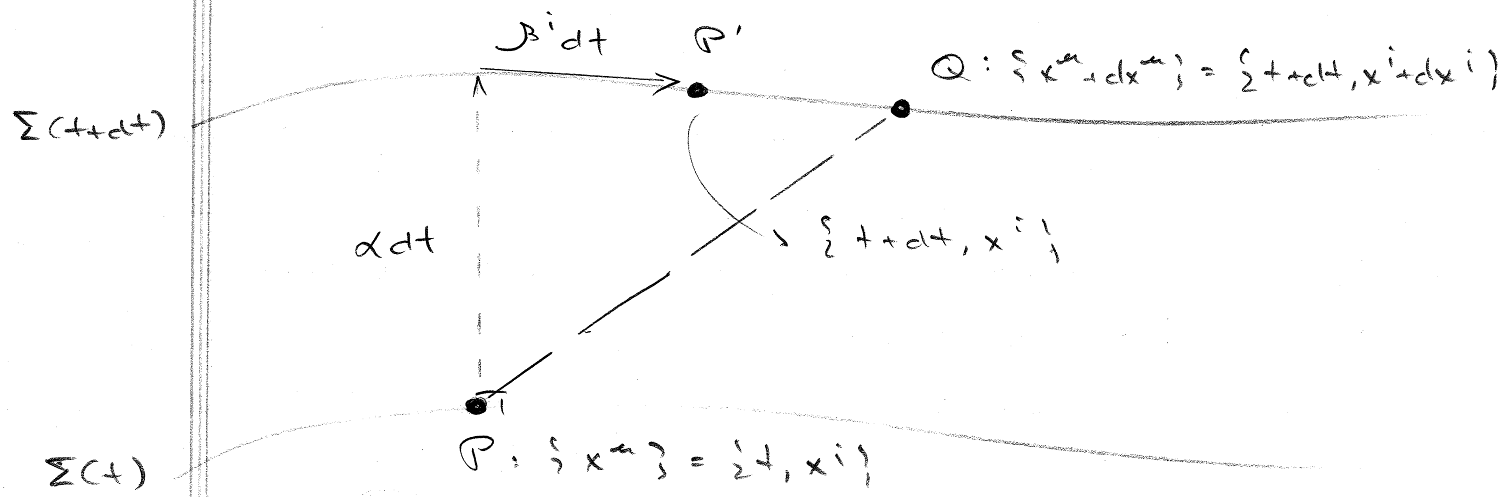
- SAFEST TO VIEW t AS TIME PARAMETER; MAY NOT (IN FACT, IN GENERAL WILL NOT) HAVE ANY PARTIC. SIGNIFICANCE AS A PHYSICAL TIME - SUCH A NOTION IS LARGELY MEANINGLESS IN GR

- EACH $\Sigma(t)$ IS A DIFFERENTIABLE MANIFOLD IN ITS OWN RIGHT, WITH A 3-METRIC ${}^{(3)}g_{ij} = {}^{(3)}g_{(ij)}$ WHICH IS INDUCED ON $\Sigma(t)$ BY THE 4-METRIC ${}^{(4)}g_{\mu\nu}$ OF THE ENVELOPING SPACETIME

- ANY GIVEN FOLIATION ($\equiv$ CHOICE of TIME COORD $\equiv$ CHOICE of SLICING) ALSO DEFINES A NATURAL, UNIT-NORM VECTOR FIELD, n^μ, WHICH IS NORMAL (ORTHOGONAL) TO THE SLICES, AND POINTS "TO THE FUTURE"

$$n^\mu n_\mu = {}^{(4)}g_{\mu\nu}\, n^\mu n^\nu = -1$$

SPACETIME DISPLACEMENT IN THE 3+1 SPLIT



"SPACETIME PYTHAGOREAN THM" ($\lim dt \to 0$) $\Rightarrow$

$$\text{distance}(P,Q)^2 = {}^{(4)}ds^2$$

$$= -\alpha^2 dt^2 + {}^{(3)}g_{ij}\,(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$= (-\alpha^2 + {}^{(3)}g_{ij}\beta^i\beta^j)\,dt^2 + 2\,{}^{(3)}g_{ij}\beta^i dx^j dt$$

$$+ {}^{(3)}g_{ij}\,dx^i dx^j \qquad (1)$$

NOMENCLATURE:

$\alpha \equiv \alpha(t, x^i)$: LAPSE FUNCTION ("LAPSE") : GIVES LAPSE OF PROPER TIME PER UNIT COORDINATE TIME FOR AN OBSERVER MOVING NORMAL TO THE SLICES

$\beta^j = \beta^j(t, x^i)$: SHIFT VECTOR ("SHIFT") : 3-VECTOR DESCRIBING SHIFTING OF SPATIAL COORDINATES RELATIVE TO "NORMAL PROPAGATION"

• TOGETHER $\{\alpha, \beta^i\}$ CONSTITUTE 4-FOLD COORD. FREEDOM of GR

DUAL VIEWS

(1) α, β^i ARE ESSENTIALLY FREELY SPECIFIABLE FCNS (COORDINATE FREEDOM)

(2) SOME PRESCRIPTION FOR α, β^i MUST BE GIVEN "FROM OUTSIDE" - I.E. E.O.M. (EINSTEIN EQ$^{\text{NS}}$) ALONE WILL NOT, IN GENERAL, DETERMINE THEM ("GAUGE FIXING")

NOTE: TENSORS SUCH AS β^i ARE DEFINED ON $\Sigma(t)$, AND ARE CALLED SPATIAL TENSORS

CLEARLY, ${}^{(3)}g_{ij}$ IS A SPATIAL TENSOR; IT HAS AN ASSOCIATED INVERSE ${}^{(3)}g^{ij}$ SATISFYING

$$ {}^{(3)}g^{ij}\,{}^{(3)}g_{jk} = \delta^i{}_k $$

INDICES ON 3-TENSORS ARE RAISED/LOWERED WITH ${}^{(3)}g^{ij}$, ${}^{(3)}g_{ij}$; THUS, WE CAN REWRITE (1) AS

$$ {}^{(4)}ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_j\,dx^j\,dt + {}^{(3)}g_{ij}\,dx^i dx^j $$

SO, WE HAVE

$$ {}^{(4)}g_{\mu\nu} = \begin{bmatrix} {}^{(4)}g_{00} & {}^{(4)}g_{0j} \\ {}^{(4)}g_{i0} & {}^{(4)}g_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha^2 + \beta^k\beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad (2) $$

IN PARTICULAR, NOTE THAT

$$ {}^{(4)}g_{ij} = {}^{(3)}g_{ij} $$

I.E. SPATIAL COVARIANT COMPONENTS OF 4- AND 3-METRICS ARE IDENTICAL

THIS IS A GENERAL RESULT; GIVEN ANY 4-TENSOR OF TYPE $(0,k)$ (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE COMPONENTS OF A TYPE $(0,k)$ 3-TENSOR

WHY? RECALL THAT COVARIANT TENSOR COMPONENTS CAN BE DEFINED IN TERMS OF THE ACTION OF THE TENSOR ON THE COORDINATE BASIS VECTORS ${}^{(4)}\vec{e}_\mu$, $\mu = 0, 1, 2, 3$

$$\text{E.G.} \quad {}^{(4)}t_{\mu\nu} = {}^{(4)}t\left({}^{(4)}\vec{e}_\mu, {}^{(4)}\vec{e}_\nu\right)$$

$$\text{AND} \quad {}^{(4)}t_{ij} = {}^{(4)}t\left({}^{(4)}\vec{e}_i, {}^{(4)}\vec{e}_j\right)$$

BUT CLEARLY THE $\{{}^{(4)}\vec{e}_i\}$ ARE PRECISELY THE COORDINATE BASIS VECTORS $\{{}^{(3)}\vec{e}_i\}$ FOR $\Sigma(t)$ WITH COORDINATES $\{x^i\}$; THUS INTERPRETING ${}^{(4)}t(\ldots)$ AS A 3-TENSOR ${}^{(3)}t(\ldots)$, WE NECESSARILY HAVE

$${}^{(3)}t_{ij} = {}^{(4)}t_{ij}$$

ON THE OTHER HAND, THE SPATIAL MEMBERS $\{{}^{(4)}\underset{\sim}{\omega}^i\}$ OF THE DUAL BASIS $\{{}^{(4)}\underset{\sim}{\omega}^\mu\}$, WILL NOT, IN GENERAL COINCIDE WITH $\{{}^{(3)}\underset{\sim}{\omega}^i\}$ – THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

- THUS, IN GENERAL

$${}^{(3)}t^{ij} \neq {}^{(4)}t^{ij}$$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$${}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3)$$

(EXERCISE: VERIFY)

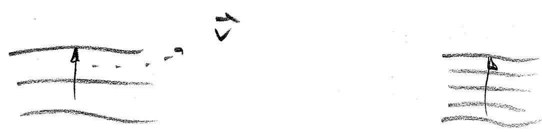
* FROM (2) WE CAN ALSO COMPUTE THE USEFUL RESULT

$$\sqrt{-{}^{(4)}g} = \alpha \sqrt{{}^{(3)}g} \qquad (4)$$

THE NORMAL VECTOR FIELD n^μ

* EASIEST TO START WITH ASSOC. DUAL-VECTOR (ONE-FORM) FIELD, n_μ

GEOM. INTERP of DUAL-VECTOR FIELD: LEVEL SURFACES of SCALAR FUNCTION → DUAL NOTION TO "INFINITESIMAL DISPLACEMENT" (VECTOR)

$\underline{d}f$  $2\,\underline{d}f$

$$\langle \vec{v}, \underline{d}f \rangle = v^\mu (df)_\mu = \text{\# of level surfaces "pierced" by } \vec{v}$$

* HERE, OUR SCALAR FUNCTION IS THE TIME COORDINATE t, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{d}t$, THEN

$$\underline{n} \propto \underline{d}t$$

OR IN COMPONENT FORM

$$n_\mu = (n_0, 0, 0, 0)$$

THEN, FROM

$${}^{(4)}g^{\mu\nu} n_\mu n_\nu = -1$$

WE HAVE

SIGN CHOSEN SO THAT n^μ IS "FUTURE-DIRECTED"

$$n_\mu = (-\alpha, 0, 0, 0) \tag{5}$$

AND THEN

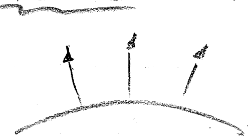
$$n^\mu = {}^{(4)}g^{\mu\nu} n_\nu = \left(\frac{1}{\alpha} , - \frac{\beta^i}{\alpha} \right) \tag{6}$$

EXTRINSIC CURVATURE

- THE INTRINSIC GEOMETRY OF $\Sigma(t)$ IS DESCRIBED BY ${}^{(3)}g_{ij}$ WHICH ENCODES ALL GEOM. INFO. WHICH MAY BE DEDUCED BY MAKING MEASUREMENTS ON $\Sigma(t)$ ALONE

- HOWEVER, A GIVEN 3-GEOMETRY (SLICE, HYPERSURFACE) MAY BE EMBEDDED IN SPACETIME IN INFINITELY MANY DISTINGUISHABLE (BY 4-D MEASUREMENTS) WAYS

EXAMPLE: 2D EMBEDDED IN 3D; FLAT SURFACE EMBEDDED WITH / WITHOUT EXTRINSIC CURVATURE



- THE MANNER IN WHICH $\Sigma(t)$ IS EMBEDDED CAN BE CHARACTERIZED BY INVESTIGATING THE CHANGE IN THE DIRECTION OF THE NORMAL FIELD AS A FCN OF POS ON $\Sigma(t)$ — THIS DEFINES THE EXTRINSIC CURVATURE TENSOR (AKA THE SECOND FUNDAMENTAL FORM)

$$K_{ij} = -\nabla_i n_j = -\nabla_{(i} n_{j)} \tag{7}$$

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij}\, n_\mu$$

BUT $n_\mu = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha\, {}^{(4)}\Gamma^{0}{}_{ij}$$

$$= \alpha\left({}^{(4)}g^{00}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0\ell}\, {}^{(4)}\Gamma_{\ell ij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2}\left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - \underline{{}^{(4)}g_{ij,0}} \right)$$

THUS

$$K_{ij} = -\frac{1}{2\alpha} \frac{\partial\, {}^{(3)}g_{ij}}{\partial t} + \ldots$$

SO WE CAN VIEW THE EXTRINSIC CURVATURE AS THE "VELOCITY" OF THE 3-METRIC ${}^{(3)}g_{ij}$ ("CONJUGATE MOMENTA")