SPHERICAL SYMMETRY (SS)

- IN SS SPACETIME NEED MATTER FOR DYNAMICS (BIRKHOFF'S THM, UNIQUENESS of SCHWARZSCHILD SOL^N AS SOL^N of $G_{ab} = 0$)

- WILL RESTRICT MATTER CONTENT TO SINGLE, MASSLESS, MINIMALLY-COUPLED SCALAR FIELD, ϕ
  - GOOD MODEL PROBLEM FOR STUDYING STRONG-FIELD, RADIATIVE S.T'S — INCLUDING BLACK HOLE FORMATION
  - EXHIBITS INTERESTING PHYSICAL BEHAVIOUR → CRITICAL PHENOMENA aka BLACK HOLE THRESHOLD PHENOMENA

- WILL REFER TO SYSTEM (SPH. SYMMETRY IMPLICIT) AS EMKG & EKG (EINSTEIN-MASSLESS KLEIN-GORDON)

LAGRANGIAN DENSITY FOR EMKG SYSTEM

$$\mathcal{L} = \mathcal{L}_{GRAV} + \mathcal{L}_\phi$$

$$= \sqrt{-g}\left(R - \tfrac{1}{2}\nabla_a\phi\,\nabla^a\phi\right)$$

"COVARIANT" E.O.M

$$G_{ab} = 8\pi T_{ab} = 8\pi\left(\nabla_a\phi\nabla_b\phi - \tfrac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\right)$$

$$\Box\phi = \nabla^a\nabla_a\phi = 0$$

3+1 FORM OF SPACETIME METRIC IN SS

- COORDINATES (t, r, θ, ϕ) ADAPTED TO S.S.

METRIC ON UNIT 2-SPHERE $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

- THEN MOST GENERAL 3-METRIC IS

$$\gamma_{ij} = \mathrm{diag}\left(a^2(r,t),\ r^2 b^2(r,t),\ r^2 b^2 \sin^2\theta\right) \qquad (1)$$

THE LAPSE FUNCTION IS $\alpha(r,t)$, AND THE SHIFT VECTOR $\beta^i(r,t)$ HAS ONLY A RADIAL COMPONENT, $\beta(r,t)$

$$\beta^i = (\beta, 0, 0) \qquad (2)$$

$$\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0) \qquad (3)$$

- THE MOST GENERAL 4-METRIC IS THEN

$$ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_i \, dt\, dx^i + \gamma_{ij}\, dx^i dx^j$$

$$= (-\alpha^2 + a^2\beta^2)\,dt^2 + 2a^2\beta \, dt\, dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \qquad (4)$$

- CORRESPONDING EXTRINSIC CURVATURE TENSOR, $K^i{}_j$, LIKE γ_{ij}, HAS ONLY TWO INDEPENDENT COMPONENTS

$$K^i{}_j = \mathrm{diag}\left(K^r{}_r(r,t),\ K^\theta{}_\theta(r,t),\ K^\theta{}_\theta\right) \qquad (5)$$

EASY TO SHOW EQUALITY FROM

$$K_{ij} = (2\alpha)^{-1}\left(-\partial_t \gamma_{ij} + D_i\beta_j + D_j\beta_i\right)$$

→ SO, HAVE REDUCED TOTAL # OF GRAVITATIONAL KIN. / DYN VBLS FROM 16 TO 6, AND, OF COURSE, THESE VBLS ARE FUNCTIONS ONLY of (r,t) RATHER THAN (x,y,z,t)

EINSTEIN EQUATIONS

(1) CONSTRAINTS

$$\boxed{\begin{aligned} &R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho && (6) \\ &D_j K_i{}^j - D_i K = 8\pi j_i && (7) \end{aligned}}$$

↳ NOTE INDEX SHIFT RELATIVE TO PREVIOUS FORM

WHERE $\rho = n_\mu n_\nu T^{\mu\nu}$ (8)

$$j_i = \gamma_{ik} j^k = -n_\mu T^\mu{}_i \qquad (9)$$

RECALL: $n_\mu = (-\alpha, 0, 0, 0)$ (10)

(2) EVOLUTION EQUATIONS $\left(\dot{} \equiv \frac{\partial}{\partial t} \equiv \partial_t \right)$

$$\boxed{\begin{aligned} \dot\gamma_{ij} &= -2\alpha\gamma_{ik}K^k{}_j + \beta^k\partial_k\gamma_{ij} + \gamma_{ik}\partial_j\beta^k + \gamma_{kj}\partial_i\beta^k && (11) \\ \dot K^i{}_j &= \beta^k\partial_k K^i{}_j - \partial_k\beta^i K^k{}_j + \partial_j\beta^k K^i{}_k - D^iD_j\alpha \\ (12)\quad &\quad + \alpha\left(R^i{}_j + KK^i{}_j + 4\pi(S-\rho)\delta^i{}_j - 8\pi S^i{}_j\right) \end{aligned}}$$

WHERE $\underset{(13)}{S_{ij} = T_{ij}}$, $\underset{(14)}{S^i{}_j = \gamma^{ik}S_{kj}}$, $\underset{(15)}{S = S^i{}_i}$

• NEED CHRISTOFFEL SYMBOLS $\Gamma^i{}_{jk}$, RICCI COMPONENTS $R^i{}_j$ AND RICCI SCALAR R ASSOCIATED WITH γ_{ij} (1). USING STANDARD FORMULAE FIND FOLLOWING NON-VANISHING $\Gamma^i{}_{jk}$ ($' \equiv \frac{\partial}{\partial r} \equiv \partial_r$)

$$\Gamma^r{}_{rr} = \frac{a'}{a} \qquad \Gamma^r{}_{\theta\theta} = \frac{-(r^2b^2)'}{2a^2} \qquad \Gamma^r{}_{\phi\phi} = \sin^2\theta\, \Gamma^r{}_{\theta\theta}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{(r^2b^2)'}{2(r^2b^2)} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \qquad (16a\text{-}g)$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \Gamma^\theta{}_{\theta r} \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$$

FROM THESE WE COMPUTE NON-VANISHING $R^i{}_j$

$$R^r{}_r = \frac{-2}{arb}\left(\frac{(rb)'}{a}\right)' \qquad (17a)$$

$$R^\theta{}_\theta = R^\phi{}_\phi = \frac{1}{a(rb)^2}\left(a - \left(\frac{rb}{a}(rb)'\right)'\right) \qquad (17b)$$

AND FINALLY, THE SCALAR CURVATURE, R, IS

$$R = R^r{}_r + R^\theta{}_\theta + R^\phi{}_\phi = R^r{}_r + 2R^\theta{}_\theta$$

$$= \frac{-2}{arb}\left(\left(\frac{(rb)'}{a}\right)' + \frac{1}{rb}\left(\left(\frac{rb}{a}(rb)'\right)' - a\right)\right) \qquad (18)$$

• IN THE EVOLUTION EQUATION FOR $K^i{}_j$ WE NEED TO EVALUATE $D^iD_j\alpha$:

$$D^i D_j \alpha = \gamma^{ik} D_k D_j \alpha = \gamma^{ik} D_k \partial_j \alpha$$

$$= \gamma^{ik} \left(\partial_k \partial_j \alpha - \Gamma^m{}_{jk} \partial_m \alpha \right)$$

USING RESULTS FROM ABOVE, WE FIND

$$D^r D_r \alpha = \frac{1}{a} \left(\frac{\alpha'}{a} \right)' \qquad (19a)$$

$$D^\theta D_\theta \alpha = D^\phi D_\phi \alpha = \frac{\alpha' (rb)'}{a^2 rb} \qquad (19b)$$

- ALSO NEED STRESS-TENSOR "COMPONENTS" ρ, j_i, $S^i{}_j$

- IN SPIRIT of HAMILTONIAN APPROACH, IT IS CONVENIENT TO INTRODUCE AUXILIARY FUNCTIONS

$$\Phi(r,t) \equiv \phi'(r,t) = \partial_r \phi(r,t) \qquad (20)$$

$$\Pi(r,t) \equiv \frac{a}{\alpha} \left(\dot\phi - \beta \phi' \right) \qquad (21)$$

VIEW Φ, Π AS "REAL" DYNAMICAL VBLS FOR SCALAR FIELD; NOTE: FOR MASSLESS FIELD, VALUE of ϕ IS MEANINGLESS ($\phi \to \phi + \text{const}$ STILL SATISFIES $\Box \phi = 0$) ALL "ACTION" IS IN GRADIENTS of ϕ (I.E. IN Φ AND Π)

ALSO NOTE THAT WE HAVE (c/f (4))

$$g_{tt} = -\alpha^2 + a^2\beta^2 \qquad g_{tr} = g_{rt} = a^2\beta$$

(22a-e)

$$g_{rr} = a^2 \qquad g_{\theta\theta} = r^2b^2 \qquad g_{\phi\phi} = r^2b^2\sin^2\theta$$

AND

$$g^{tt} = -\alpha^{-2} \qquad g^{tr} = g^{rt} = \beta\alpha^{-2} \qquad (23a-e)$$

$$g^{rr} = a^{-2} - \beta^2\alpha^{-2} \qquad g^{\theta\theta} = (rb)^{-2} \qquad g^{\phi\phi} = (rb\sin\theta)^{-2}$$

THEN WE FIND

$$\nabla^t\phi \equiv \phi^{,t} = g^{tt}\partial_t\phi + g^{tr}\partial_r\phi = -\frac{\Pi}{\alpha a} \qquad (24)$$

$$(\nabla^\mu\phi)(\nabla_\mu\phi) = \phi^{,\mu}\phi_{,\mu} = \frac{\Phi^2 - \Pi^2}{a^2} \qquad (25)$$

AND, AGAIN RECALLING THAT $n_\mu = (-\alpha, 0, 0, 0)$, WE FIND

$$\rho = n_\mu n_\nu T^{\mu\nu} = \alpha^2 T^{tt} = \alpha^2\left(\phi^{,t}\phi^{,t} - \tfrac{1}{2}g^{tt}\phi^{,\mu}\phi_{,\mu}\right)$$

$$= \frac{\Phi^2 + \Pi^2}{2a^2} \qquad (26)$$

$$j_i = (j_r, 0, 0)$$

$$j_r = -n_\mu T^\mu{}_i = \alpha T^0{}_i = \alpha\,\phi^{,t}\phi_{,r} = -\frac{\Phi\Pi}{a} \qquad (27)$$

FINALLY, WE HAVE THE SPATIAL STRESS COMPONENTS

$$S^i{}_j = \gamma^{ik} S_{kj} = \gamma^{ik} T_{kj}$$

FROM WHICH WE FIND

$$S^r{}_r = \rho = \frac{\Phi^2 + \Pi^2}{2a^2} \qquad (28)$$

$$S^\theta{}_\theta = S^\phi{}_\phi = \frac{\Pi^2 - \Phi^2}{2a^2} \qquad (29)$$

$$S - \rho = 2 S^\theta{}_\theta = \frac{\Pi^2 - \Phi^2}{a^2} \qquad (30)$$

• WE CAN NOW ASSEMBLE THE ABOVE RESULTS TO PRODUCE THE SPHERICALLY-SYMMETRIC SPECIALIZATION of THE GENERAL 3+1 EQUATIONS (6), (7), (11) ; (12)

A) HAMILTONIAN CONSTRAINT

$$R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho$$

$$-K^i{}_j K^j{}_i + K^2 = -\left(K^r{}_r{}^2 + 2K^\theta{}_\theta{}^2\right) + \left(K^r{}_r + 2K^\theta{}_\theta\right)^2$$

$$= 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2$$

$$\boxed{R + 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2 = 8\pi \frac{\Phi^2 + \Pi^2}{a^2}} \qquad (31)$$

SEE (18)

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NONTRIVIAL)

- FIRST NOTE THAT

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

$$= K^r{}_r{}' + 2\Gamma^\theta{}_{r\theta}\left(K^r{}_r - K^\theta{}_\theta\right)$$

$$D_r K = \left(K^r{}_r + 2K^\theta{}_\theta\right)'$$

THEN WE HAVE FROM (7) AND (27)

$$\boxed{K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = 4\pi \frac{\Phi \Pi}{a}} \qquad (32)$$

C) EVOLUTION EQUATIONS FOR γ_{ij} (a, b)

- FOLLOW DIRECTLY FROM (11), RECALL, CAN BE VIEWED AS DEF^{N} OF $K^i{}_j$

$$\dot{a} = -\alpha a K^r{}_r + (a\beta)' \qquad (33)$$

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' \qquad (34)$$

D) EVOLUTION EQUATIONS FOR $K^i{}_j$ $(K^r{}_r, K^\theta{}_\theta)$

- FOLLOW DIRECTLY FROM (12) AND OTHER RESULTS ABOVE

$$\dot{K}^r{}_r = \beta K^r{}_r{}' - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left(-\frac{2}{rab}\left(\frac{(rb)'}{a}\right)' + K K^r{}_r - 8\pi \frac{\Phi^2}{a^2}\right) \qquad (35)$$

$$\dot{K}^\theta{}_\theta = \beta K^\theta{}_\theta{}' + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left(\frac{\alpha rb}{a}(rb)'\right)' + \alpha K K^\theta{}_\theta \quad (36)$$

- (31)-(36) ARE THE COMPLETE 3+1 EQUATIONS FOR THE GEOMETRIC VARIABLES (NOTE: WE HAVE SAID NOTHING YET RE COORDINATE CHOICES, I.E. RE SPECIFICATIONS OF α AND β)

MASSLESS KLEIN GORDON EQUATION

- NEED E.O.M. FOR Φ AND Π
- RECALL DEF^N OF Π, (21)

$$\Pi \equiv \frac{a}{\alpha}\left(\dot\phi - \beta\phi'\right)$$

$$\rightarrow \quad \dot\phi = \frac{\alpha}{a}\Pi + \beta\phi' = \frac{\alpha}{a}\Pi + \beta\Phi$$

- BUT $\dot\phi' = \dot\Phi$, SO

$$\boxed{\dot\Phi = \left(\beta\Phi + \frac{\alpha}{a}\Pi\right)' \quad (37)}$$

- TO FIND $\dot\Pi$ EQN, RECALL THAT

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right)$$

$$\Box\phi = 0 \rightarrow \partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) = 0$$

$$\rightarrow \partial_t\left(\sqrt{-g}\, g^{\nu t}\partial_\nu \phi\right) + \partial_r\left(\sqrt{-g}\, g^{\nu r}\partial_\nu \phi\right)$$

$$= \partial_t\left(\alpha a r^2 b^2\left(-\alpha^{-2}\dot{\phi} + \beta\alpha^{-2}\phi'\right)\right)$$

$$+ \partial_r\left(\alpha a r^2 b^2\left(\left(a^{-2} - \beta^2\alpha^{-2}\right)\phi' + \beta\alpha^{-2}\dot{\phi}\right)\right)$$

$$= 0$$

$$\rightarrow \dot{\left(r^2 b^2 \Pi\right)} = \left(r^2 b^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right)'$$

WRITE AS $(r^2b^2)\mathring{\Pi} + \mathring{(r^2b^2)}\,\Pi$ AND USE EVOLUTION EQUATION (39) FOR $\mathring{b}$

$$\boxed{\mathring{\Pi} = \frac{1}{r^2b^2}\left(r^2b^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right)' + 2\left(\alpha K^\theta{}_\theta - \beta\frac{(rb)'}{rb}\right)\Pi} \qquad (38)$$

CHARACTERISTIC ANALYSIS of THE SCALAR FIELD

REF: COURANT & HILBERT "METHODS of MATH. PHYS", VOL II, ch 5

- EQN^S (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u \equiv (\Phi, \Pi)^T$$

WE CAN WRITE

$$u_t + A u_x = B \qquad (39)$$

$$A = -\begin{pmatrix} \beta & \alpha/a \\ \alpha/a & \beta \end{pmatrix} \qquad (40)$$

AND B IS ANOTHER MATRIX WHICH DOES NOT INVOLVE DERIVATIVES of u.

- THE CHARACTERISTIC DIRECTIONS $\tau \equiv dr/dt$ OF (39)-(40) ARE GIVEN BY

$$|A - \tau \mathbb{1}| = 0$$

$$\rightarrow \quad \boxed{\tau = -\beta \pm \frac{\alpha}{a}} \qquad (41)$$

THESE ARE THE "LOCAL SIGNAL SPEEDS" FOR THE SCALAR FIELD

• MASSLESS SCALAR FIELD — WEAK (NON-SELF GRAVITATING)

SIGNALS TRAVEL ALONG NULL GEODESICS → ALTERNATE DERIVATION of (41)

$$ds^2 = -\alpha^2 dt^2 + a^2 (dr + \beta dt)^2 = 0$$

REGULARITY / LOCAL FLATNESS AT $r=0$

• OUR CAUCHY PROBLEM FOR THE EMKG MODEL IS TO BE SOLVED ON

$$t \geq 0 \,, \quad r \geq 0$$

• BOUNDARY CONDITIONS AS $r \to \infty$ WILL FOLLOW FROM ASYMPTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE DIFFERENCING) IS EFFECTIVELY ONE

• GET CONDITIONS AT $r=0$ BY DEMANDING THAT SCALAR & GRAV. FIELDS BE REGULAR, AND THAT THE S.T. BE LOCALLY FLAT THERE

• TRICKY ISSUE IN GENERAL UNLESS WE MAKE ASSUMPTIONS ABOUT SLICING, SPATIAL COORDINATES (SEE BARDEEN & PIRAN, PHYSICS REPORTS 96 (1983) 205-250); WE WILL ASSUME EVERYWHERE SMOOTH SLICINGS AND SPATIAL COORDS

(A) SCALAR FIELD

$$\nabla \phi = \phi' \hat{r}$$

NOT DEFINED AT $r=0$ UNLESS

$$\Phi'(0,t) = 0 \qquad (42)$$

(3) GRAVITATIONAL FIELD $\rightarrow \Phi(0,t) = \Phi_0(t) + r^2\Phi_2(t) + O(r^4)$

- BARDEEN ; PIRAN: REGULARITY ≡ ALL TENSOR COMPONENTS CAN BE EXPANDED IN NON-NEG. POWERS OF X, Y, Z:

$$X = r\sin\theta\cos\varphi \qquad Y = r\sin\theta\sin\varphi \qquad Z = r\cos\theta$$

- APPLYING TO OUR CASE (WITH ASSUMPTION of SMOOTH COORDS) FIND

$$g(0,t) = g_0(t) + r^2 g_2(t) + \ldots$$

WHERE g IS ANY OF $a, b, K^r{}_r$ OR $K^\theta{}_\theta$

→ $$a'(0,t) = b'(0,t) = K^r{}_r{}'(0,t) = K^\theta{}_\theta{}'(0,t) = 0$$
(43a-d)

- THESE CONDITIONS WILL BE CONSISTENT WITH EVOL. EQNS ONLY IF

$$\alpha'(0,t) = \beta(0,t) = 0 \qquad (44)$$

$$\beta(0,t) = r\beta_1(t) + O(r^3)$$

(SMOOTHNESS of COORDINATES)

- LOCAL FLATNESS : CONSIDER ∥ TRANSPORT OF AN ARBITRARY VECTOR ABOUT CLOSED LOOP ENCLOSING

$r \to 0$ ON $\theta = \frac{\pi}{2}$ (EQUATORIAL PLANE); DEMAND THAT THERE BE NO NET ROTATION IN LIMIT LOOP SHRUNK TO POINT $\equiv \lim_{r \to 0}$ PROPER CIRCUM / PROPER RADIUS $= 2\pi$ AND

$$\nabla^{\mu}(rb)\nabla_{\mu}(rb) = 1$$

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS FIND

$$a(0,t) = b(0,t) \qquad (45)$$

THIS + $\dot{a}, \dot{b}$ EQNS + REG. THEN IMPLY

$$K^{r}{}_{r}(0,t) = K^{\theta}{}_{\theta}(0,t) \qquad (46)$$

- NOTE: REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

TRAPPED SURFACES / APPARENT HORIZONS

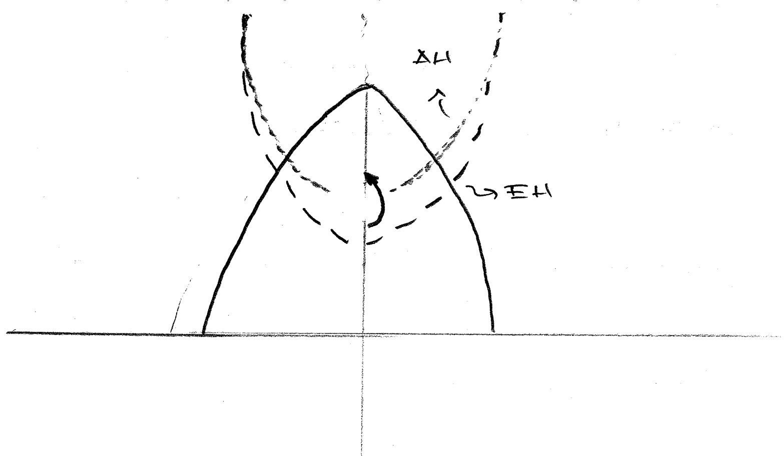
- WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

- USEFUL TO BE ABLE TO COMPUTE "INSTANTANEOUS" APPROX. (I.E. ON ANY HYPERSURFACE $\Sigma(t)$) TO EH. — PROVIDED BY APPARENT HORIZON $\equiv$ OUTERMOST MARGINALLY TRAPPED SURFACE

- TRAPPED SURFACE: 2-SURFACE WITH TOPOLOGY S^2 SUCH THAT DIVERGENCE OF OUTGOING NULL GEODESICS EMANATING FROM SURFACE < 0

- MARGINALLY TRAPPED SURFACE: "< 0" → "$= 0$"

- MODULO COSMIC CENSORSHIP (NO NAKED SINGULARITIES) EXISTENCE OF AH ⇒ EXISTENCE OF EH; HOWEVER CAN HAVE EH WITHOUT AH



- WILL TEND TO USE TERMS AH, B : MTS INTER-CHANGABLY, BUT SHOULD BE AWARE OF DISTINCTIONS

(MARGINALLY) TRAPPED SURFACE EQN (AH EQN)

- CONSIDER A 2-SURFACE WITH OUTGOING NULL TANGENT u^a WHICH IS MARGINALLY TRAPPED, THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

$$u^a = s^a + n^a$$

↳ UNIT FUTURE-DIRECTED TIMELIKE NORMAL TO Σ

↳ UNIT OUTWARDS-POINTING SPACELIKE NORMAL TO 2-SURF.

IN // TO 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED ON THE 2-SURFACE BY PROJECTION

$$h^{ab} = \gamma^{ab} - s^a s^b = g^{ab} + n^a n^b - s^a s^b$$

CAN SHOW (EXERCISE) ~~THAT~~ $\nabla_a u^a$ IS A "2-TENSOR"; I.E. IS INTRINSIC TO 2-SURFACE; I.E. DOES NOT DEPEND ON HOW 2-SURFACE IS EMBEDDED IN Σ, THEN

$$\nabla_a u^a = g^{ab} \nabla_a u_b = h^{ab} \nabla_a u_b$$

$$= h^{ab} \nabla_a (s_b + n_b)$$

$$= h^{ab} \perp \nabla_a (s_b + n_b)$$

h_{ab} PROJECTS ONTO 2-SURFACE, SO CAN FIRST PROJECT ONTO Σ

$$= h^{ab} (D_a s_b + \perp \nabla_a n_b)$$

$$= h^{ab} (D_a s_b - K_{ab})$$

$$= (g^{ab} - s^a s^b)(D_a s_b - K_{ab})$$

$$= D^a s_a - K + s^a s^b K_{ab}$$

$$\left(s^b D_a s_b = \tfrac{1}{2} D_a (s^b s_b) = \tfrac{1}{2} D_a (1) = 0 \right)$$

THUS, OUR TRAPPED SURFACE (AH) EQUATION IS

$$\boxed{D^a s_a - K + s^a s^b K_{ab} = 0} \qquad (47)$$

AND ARGUING AS WE DID FOR THE 3+1 EQNS WE GET A VALID COMPONENT FORM OF THIS EQUATION BY TAKING $a \to i$, $b \to j$

$$\boxed{D^i s_i - K + s^i s^j K_{ij} = 0} \qquad (48)$$

SPECIALIZING NOW TO SPHERICAL SYMMETRY

$$ds^2 = a^2 dr^2 + r^2 b^2 d\Omega^2$$

$$\gamma_{ij} s^i s^j = 1 \to s^i = (a^{-1}, 0, 0)$$

$$D_i s^i = \gamma^{-\frac{1}{2}} \partial_i (\gamma^{\frac{1}{2}} s^i) \qquad \gamma^{\frac{1}{2}} = a r^2 b^2$$

$$= \frac{1}{a r^2 b^2} (r^2 b^2)' = \frac{2 (rb)'}{arb}$$

THUS, (48) BECOMES

$$\frac{2(rb)'}{arb} - (K^r{}_r + 2K^\theta{}_\theta) + \underbrace{a^{-2} K_{rr}}_{K^r{}_r} = 0$$

$$\boxed{(rb)' = arb\, K^\theta{}_\theta} \qquad (49)$$

NOW, RECALL EVOL. EQN (39) FOR b

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)'$$

$$\Rightarrow K^\theta{}_\theta = -(\alpha b)^{-1}\left(\dot{b} - \frac{\beta}{r}(rb)'\right)$$

$$rK^\theta{}_\theta = -(\alpha b)^{-1}\left((r\dot{b}) - \beta(rb)'\right)$$

SO (49) CAN BE REWRITTEN AS

$$\boxed{(r\dot{b}) + \left(\frac{\alpha}{a} - \beta\right)(rb)' = 0} \qquad (50)$$

WHICH SAYS THAT THE SURFACE OF CONSTANT AREAL RADIUS $R = rb$ IS OUTGOING NULL AT THE MARGINALLY TRAPPED SURFACE IN ACCORD WITH OUR PHYSICAL PICTURE

COORDINATE CONDITIONS FOR S.S. SYSTEMS

• NOT EXHAUSTIVE, WILL COVER MOST of "COMMON" CHOICES

(A) SLICING CONDITIONS

(i) VIA CONDITIONS ON $\mathrm{Tr}\,K = K = K^i{}_i$

TAKE TRACE of (12) (IN S.S.)

$$\dot{K} = \beta K' - D^i D_i \alpha + \alpha\left(\mathcal{R} + K^2 + 4\pi(S - 3\rho)\right)$$

CAN ELIMINATE $\mathcal{R}$ (EXPENSIVE TO EVALUATE, NOT SO CRUCIAL HERE) USING HAMILTONIAN CONSTRAINT

$$R = K^i{}_j K^j{}_i - K^2 + 16\pi\rho$$

$$\rightarrow \quad \dot{K} = \beta K' - D^i D_i \alpha + \alpha\left(K^i{}_j K^j{}_i + 4\pi(S+\rho)\right) \qquad (51)$$

(ia) MAXIMAL SLICING $K = 0$ (LICHNEROWICZ)

• IMPLEMENT BY CHOOSING COMPATIBLE $\Sigma(0)$

$$K(r,0) = 0$$

THEN IMPOSE

$$\dot{K}(r,t) = 0 \qquad t > 0$$

THEN (51) CAN BE VIEWED AS ELLIPTIC EQN FOR LAPSE

$$\boxed{D^i D_i \alpha = \alpha\left(K^i{}_j K^j{}_i + 4\pi(S+\rho)\right)} \qquad (52)$$

(ib) POLAR SLICING $K = K^r{}_r$ (BARDEEN; PIRAN)

$$K = K^r{}_r + 2K^\theta{}_\theta = K^r{}_r \rightarrow K^\theta{}_\theta = 0$$

• AGAIN, IMPLEMENT BY CHOOSING $\Sigma(0)$ SO THAT

$$K^\theta{}_\theta(r,0) = 0$$

THEN DEMAND

$$\dot{K}^{\theta}{}_{\theta}(r,t) = 0 \qquad t \geq 0$$

* RECALL EQN (36) FOR $\dot{K}^{\theta}{}_{\theta}$

$$\dot{K}^{\theta}{}_{\theta} = \beta K^{\theta}{}_{\theta}{}' + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left(\frac{\alpha rb}{a}(rb)'\right)' + \alpha K K^{\theta}{}_{\theta}$$

USING $K^{\theta}{}_{\theta} = \dot{K}^{\theta}{}_{\theta} = K^{\theta}{}_{\theta}{}' = 0$, THIS BECOMES A FIRST-ORDER HOMOGENEOUS ODE FOR α

$$\boxed{\left(\frac{\alpha rb}{a}(rb)'\right)' - \alpha a = 0} \qquad (53)$$

(ii) INGOING EDDINGTON-FINKELSTEIN TYPE

* DEMAND THAT t BE CHOSEN SO THAT $\frac{\partial}{\partial t}^{a} - \frac{\partial}{\partial r}^{a}$ IS NULL

$$g_{\mu\nu}\left(\frac{\partial}{\partial t}^{\mu} - \frac{\partial}{\partial r}^{\mu}\right)\left(\frac{\partial}{\partial t}^{\nu} - \frac{\partial}{\partial r}^{\nu}\right) = 0$$

$$g_{tt} - 2g_{rt} + g_{rr} = 0$$

$$(-\alpha^2 + a^2\beta^2) - 2a^2\beta + a^2 = 0$$

$$-\alpha^2 + a^2(1-\beta)^2 = 0$$

ASSUMING $\beta < 1$

$$\boxed{\alpha = a(1-\beta)} \qquad (54)$$

("ALGEBRAIC CONDITION")

• NOTE THAT WE COULD ALSO HAVE DERIVED (54) FROM OUR PREVIOUS CALC. OF CHAR. DIRECTIONS

$$-1 = \frac{dr}{dt}_{\text{ingoing}} = -\beta - \frac{\alpha}{a} \rightarrow \alpha = a(1-\beta)$$

(B) SPATIAL COORDINATE CONDITIONS

(i) NORMAL COORDINATES: $\beta = 0$

(ii) AREAL COORDINATE

• DEMAND THAT r MEASURE PROPER SURFACE AREA $\rightarrow b(r,t) = 1$; AS USUAL, IMPLEMENT VIA $b(r,0) = 1$, $\overset{\circ}{b}(r,t) = 0$, RECALL $\overset{\circ}{b}$ EQN

$$\overset{\circ}{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)'$$

SO THE SHIFT COMPONENT MUST SATISFY

$$\boxed{\beta = \alpha r K^\theta{}_\theta} \qquad (55)$$

(iii) ISOTROPIC (SOMETIMES ISOTHERMAL) COORDINATES

• DEMAND THAT 3-METRIC BE CONFORMALLY FLAT

$$ds^2 = a^2(dr^2 + r^2 d\Omega^2)$$

ONCE AGAIN, IMPLEMENT BY SPECIFYING

$a(r,t) = b(r,t)$ THEN DEMAND

$$\dot{a}(r,t) = \dot{b}(r,t)$$

$$\dot{a} = -\alpha a K^r{}_r + (a\beta)'$$

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)'$$

EQUATING THE RHS'S AND USING $b = a$

$$-\alpha a K^r{}_r + a'\beta + a\beta' = -\alpha a K^\theta{}_\theta + \beta a' + \frac{\beta}{r} a$$

$$\beta' - \frac{\beta}{r} = \alpha (K^r{}_r - K^\theta{}_\theta)$$

$$\boxed{\left(\frac{\beta}{r}\right)' = \frac{\alpha(K^r{}_r - K^\theta{}_\theta)}{r}} \qquad (56)$$

$\Leftarrow$ EXAMPLE of GENERAL CLASS of CONDITIONS CALLED "MINIMAL DISTORTION" (YORK & Ó MURCHADHA (1974), BUL. AM. PHYS. SOC. **19**, 509; SEE ALSO ARTICLE IN "SOURCES..." SMARR ed.), SO CALLED SINCE THEY MINIMIZE "SHEARING" of COORDINATE ELLIPSOIDS IN MOVING FROM $\Sigma(t) \to \Sigma(t+dt)$

MINIMAL DISTORTION EQ^N

$$(\Delta_\ell \beta)^i = 2 D_j \left(\alpha \left(K^{ij} - \tfrac{1}{3}\gamma^{ij} K\right)\right) \qquad (57)$$

WHERE $(\Delta_\ell \beta)^i \equiv D_j (\ell\beta)^{ij}$ IS THE VECTOR LAPLACIAN DEFINED PREVIOUSLY IN THE IVP DISCUSSION

EMKG EQN'S IN SOME SPECIFIC COORDINATE SYSTEMS

* A) POLAR / AREAL (POLAR/RADIAL, GEN. of SCHWARZ)

B) MAXIMAL / ISOTROPIC

(C) MAXIMAL / AREAL)

D) IEF

A) POLAR / AREAL

i) α: $K = K^i{}_i = K^r{}_r$ POLAR SLICING

ii) β: $b = 1$ $(rb = r)$ AREAL COORDINATE

- RECALL SINCE $K = K^r{}_r + 2K^\theta{}_\theta$; $K = K^r{}_r \Rightarrow K^\theta{}_\theta = 0$

- AREAL CONDITION FOR SHIFT (55) (OBTAINED FROM $\dot{b} = 0$)

$$\beta = \alpha r K^\theta{}_\theta \Rightarrow \boxed{\beta = 0} \quad (58)$$

- THUS THE 4-METRIC IS DIAGONAL IN THIS CASE (ALTERNATE DERIVATION of POLAR SLICING EQN IN SPH. SYMM. $b = 1$, $\beta = 0$)

$$ds^2 = -\alpha^2(r,t)\,dt^2 + a^2(r,t)\,dr^2 + r^2 d\Omega^2 \quad (59)$$

$$K^i{}_j = \mathrm{diag}(K^r{}_r, 0, 0) \quad (60)$$

⇒ HAVE CONSIDERABLE SIMPLIFICATION of EQ^NS OF MOTION

(MAJOR MOTIVATION FOR COMPUTATIONAL USE)

CONSTRAINT EQUATIONS

i) HAMILTONIAN CONSTRAINT

$$\mathcal{R} + 4K^r{}_r K^\theta{}_\theta + 2{K^\theta{}_\theta}^2 = 16\pi\rho$$

$$\rightarrow \quad \mathcal{R} = 16\pi\rho$$

FROM GENERAL EXPRESSION FOR $\mathcal{R}$ WITH $b=1$

$$\boxed{\mathcal{R} = \frac{4}{r}\frac{a'}{a^3} + \frac{2}{r^2}\left(1 - a^{-2}\right)} \qquad (61)$$

NOW, IN ANALOGY WITH VACUUM SCHWARZ. SOL^N. DEFINE MASS ASPECT FCN, $m(r,t)$ VIA

$$\boxed{\left(1 - \frac{2m(r,t)}{r}\right)^{-1} = a(r,t)^2} \qquad (62)$$

OR

$$\boxed{m = \frac{1}{2} r \left(1 - a^{-2}\right)} \qquad (63)$$

THEN AN EASY CALCULATION SHOWS THAT

$$\boxed{m' = \frac{dm}{dr} = \frac{1}{4} r^2 \mathcal{R}} \qquad (64)$$

THUS, THE HAMILTONIAN CONSTRAINT IN THE P/A SYSTEM, FOR A GENERAL SPHERICALLY SYMMETRIC SOURCE CAN BE WRITTEN

$$\boxed{\frac{dm}{dr} = 4\pi r^2 \rho} \tag{61}$$

- HOWEVER, AS WE DISCUSSED LAST TERM, THE FAMILIAR (INTUITIVE) APPEARANCE of THIS EQN IS LARGELY A CONSEQUENCE of OUR SPECIFIC CHOICE of COORDINATES

- MASS ASPECT IS PHYSICALLY SIGNIFICANT, THOUGH SINCE WHERE $T_{ab} = 0$, OUR S.T. MUST BE A PIECE of SCHWARZSCHILD

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

I.E. WHERE $T_{ab} = 0$ WE MUST HAVE $m(r,t) = \text{constant} = \underline{M}$ = TOTAL GRAVITATING MASS CONTAINED WITHIN SPHERE of RADIUS r AT TIME t. WHERE $T_{ab} \neq 0$ ($\rho \neq 0$), INTERPRETATION of $m(r,t)$ NOT SO CLEAR CUT, BUT FOR DIAGNOSTIC PURPOSES, THERE IS NO HARM IN REGARDING IT AS A "MASS"

- AS WRITTEN, (61) NOT ENTIRELY CONVENIENT FOR NUMERICAL WORK - ρ WILL GENERALLY DEPEND IMPLICITLY ON $m(a)$ - IN EMKG CASE

$$\rho = \frac{1}{2} a^{-2} \left(\Phi^2 + \Pi^2\right)$$

BETTER TO WRITE AS EXPLICIT, NON-LINEAR ODE FOR a

$$R = 16\pi\rho \Rightarrow \frac{4}{r}\frac{a'}{a^3} + \frac{2}{r^2}\left(1 - a^{-2}\right) = 16\pi\rho$$

$$\left(\times \frac{a^2 r}{4}\right) \qquad \boxed{\frac{a'}{a} + \frac{a^2-1}{2r} - 4\pi r a^2 \rho = 0} \qquad (62)$$

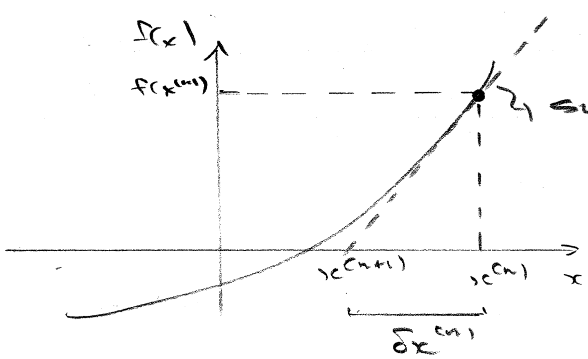
OR FOR THE EMKG SYSTEM

$$\boxed{\frac{a'}{a} + \frac{a^2-1}{2r} - 2\pi r\left(\Phi^2 + \Pi^2\right) = 0} \qquad (63)$$

SOLUTION OF (63) AND LIKE USING $O(h^2)$ F.D. AND NEWTON'S METHOD

° RECALL NEWTON'S METHOD FOR SINGLE NON-LINEAR EQN $f(x) = 0$ IN SINGLE UNKNOWN x

SEEK x^* SATISFYING $f(x^*) = 0$ ITERATIVELY, I.E. START WITH INITIAL ESTIMATE $x^{(0)}$, THEN GENERATE ITERATES $x^{(1)}, x^{(2)}, \ldots, x^{(n)}, x^{(n+1)}, \ldots$ SUCH THAT $\lim_{n\to\infty} x^{(n)} = x^*$



SLOPE $= \left.\frac{df}{dx}\right|_{x=x^{(n)}} = f'(x^{(n)})$

TERMINOLOGY $r^{(n)} \equiv f(x^{(n)})$

↳ RESIDUAL

$\{ r^{(n)} \to 0$ AS $n \to \infty \}$

$$\boxed{x^{(n+1)} = x^{(n)} - \delta x^{(n)} \quad \text{WHERE} \quad \delta x^{(n)} = \frac{f(x^{(n)})}{f'(x^{(n)})} = \frac{r^{(n)}}{f'(x^{(n)})}}$$

* STOPPING CRITERIA: TYPICALLY ITERATE UNTIL

1a) $\frac{|\delta x^{(n)}|}{|x^{(n)}|} \le \epsilon_{\delta x}$ → USER SPECIFIED THRESHOLD
TYPICALLY WANT << F.D. SOLUTION ERROR $|u^h - u| / |u|$

or

1b) $|r^{(n)}| \le \epsilon_r$

u

TRUNCATION ERROR $L^h u$
WHERE Lu IS D.E.

OR

2) DIVERGENCE DETECTED: $\delta x^{(n+2)} > \delta x^{(n+1)} > \delta x^{(n)}$
WILL GENERALLY SUFFICE

3) $n > n_{max}$ → USER SPECIFIED (50).
(TYPICALLY WILL REQUIRE MANY FEWER)

* NOTE: WHEN IT CONVERGES, NEWTON'S METHOD CONVERGES QUADRATICALLY, I.E.

$$\lim_{n\to\infty} \frac{r^{(n+1)}}{r^{(n)}} = \lim_{n\to\infty} \frac{\delta x^{(n+1)}}{\delta x^{(n)}} = 2$$

SHOULD ALWAYS EXPECT/DEMAND THIS BEHAVIOR IN PRACTICE SINCE WILL GENERALLY ONLY BE ACHIEVED IF BOTH $f(x^{(n)})$, $f'(x^{(n)})$ COMPUTATIONS ARE CORRECT

END NEWTON'S METHOD ASIDE

* A NOT IDEAL WAY COMPUTATIONALLY $a^2 - 1$ ETC CAN LEAD TO "CATASTROPHIC LOSS OF PRECISION"

FOR $a \simeq 1$ (NEAR FLAT-SPACE!!). RECAST (6?) IN NEW VBL, A

$$\boxed{A \equiv \ln a \qquad a = e^{A}}$$

WE THEN HAVE

$$\boxed{A' + \frac{e^{2A} - 1}{2r} - 2\pi r\left(\Phi^2 + \Pi^2\right) = 0} \qquad (6A)$$

DISCRETIZATION: DEFINE DISCRETE OPERATORS Δ^r_+, μ^r_+ VIA

$$\Delta^r_+ f(r) \equiv (\Delta r)^{-1}\left(f(r+\Delta r) - f(r)\right)$$

$$= f'\left(r + \frac{\Delta r}{2}\right) + O(\Delta r^2)$$

$$\mu^r_+ f(r) \equiv \frac{1}{2}\left(f(r+\Delta r) + f(r)\right)$$

$$= f\left(r + \frac{\Delta r}{2}\right) + O(\Delta r^2)$$

AND DEFINE μ^r_+ (UNLIKE RNPL OPERATORS! WARNING!!) TO HAVE PRECEDENCE OVER ALL ALGEBRAIC & FUNCTIONAL OPERATIONS; E.G. SO

$$\mu^r_+\left(\frac{f^2 g}{h}\right) = \frac{\left[\mu^r_+(f)\right]^2 \left(\mu^r_+ g\right)}{\left(\mu^r_+ h\right)}$$

THEN THE FOLLOWING IS AN $O(h^2) = O(\Delta r^2)$ DISCRETIZATION of (6A)

$$\boxed{\Delta^r_+ A + \frac{e^{2(\mu^r_+ A)} - 1}{2r} - 2\pi\, \mu^r_+\left(r\left(\Phi^2 + \Pi^2\right)\right) = 0} \qquad (6B)$$

OR USING CONVENTIONAL INDEX NOTATION (NOTE: WE'RE ASSUMING A UNIFORM SPATIAL MESH $r_j = j\,\Delta r$)

$$\boxed{(\Delta r)^{-1}\left(A_{j+1} - A_j\right) + \frac{e^{A_{j+1}+A_j}}{2 r_{j+\frac{1}{2}}} - 2\pi r_{j+\frac{1}{2}}\left(\Phi^2_{j+\frac{1}{2}} + \Pi^2_{j+\frac{1}{2}}\right) = 0} \qquad (65')$$

WHERE, E.G., $\Phi_{j+\frac{1}{2}} \equiv \frac{1}{2}\left(\Phi_{j+1} + \Phi_j\right)$

(65') IS OF THE FORM $f(A_{j+\frac{1}{2}}) = 0$, SO CAN SOLVE VIA NEWTON'S METHOD OUTLINED ABOVE WITH

$$\boxed{f'(A^{(n)}_{j+\frac{1}{2}}) = (\Delta r)^{-1} + \frac{e^{A_j + A_{j+1}}}{2 r_{j+\frac{1}{2}}}} \qquad (66)$$

- INTEGRATE (65') OUTWARDS FROM $r=0$ STARTING WITH THE INITIAL CONDITION (ASSUMING $r_1 = 0$)

$$A_1 = 0$$

WHICH FOLLOWS FROM ELEMENTARY FLATNESS AT $r=0$. $a(0,t) = b(0,t) = 1 \Rightarrow A(0,t) = 0$, THEN SOLVING EACH OF $f(A_{j+\frac{1}{2}}) = 0$, $j = 1, \cdots, nr-1$ IN TURN USING NEWTON'S METHOD ("POINT-WISE NEWTON ITERATION") NOTE: INWARDS INTEGRATION IS (NUMERICALLY) UNSTABLE. FINALLY, SET $a_j = \exp A_j$

- ONE LAST COMMENT ON HAMILTONIAN CONSTRAINT FOR EMKG:

SUGGESTIONS FOR INITIAL GUESSES.
(1) $A^{(0)}_{j+1} = A_j$
(2) $A^{(0)}_{j+1} = 2A_j - A_{j-1}$
(3) DISCRETIZE (64) USING A_j IN "SOURCE TERMS"
(4) USE $\dot{a}$ EQUATION

$$\frac{dm}{dr} = 4\pi r^2 \rho = 2\pi r^2 \left(\frac{\Phi^2 + \Pi^2}{a^2} \right) \geq 0$$

→ HAVE EASY ~~DEMONSTRATION~~ OF POSITIVITY OF GRAV. MASS IN THIS SYSTEM

ii) MOMENTUM CONSTRAINT

GENERAL:

$$\boxed{K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = -4\pi j_r} \qquad (67)$$

SO WITH $K^\theta{}_\theta = 0$, $b = 1$ THIS BECOMES AN ALGEBRAIC (!!) EQN FOR $K^r{}_r$

$$\boxed{K^r{}_r = 4\pi r j_r} \qquad (68)$$

AND FOR THE EMKG SYSTEM

$$\boxed{K^r{}_r = -4\pi r \frac{\Phi \Pi}{a}} \qquad (69)$$

INITIAL DATA

• IT IS NOW EASY TO SEE HOW AN ad hoc PROCEDURE FOR DETERMINING INITIAL DATA PROCEEDS IN THE MR SYSTEM. FROM (63) AND (69) WE SEE THAT WE CAN SIMPLY SPECIFY

$$\Phi(r,0), \quad \Pi(r,0)$$

FREELY, THEN SOLVE (63) FOR a, $K^r{}_r$ IS IMMEDIATELY GIVEN BY (69)

- FIRST NEED TO WRITE DOWN MORE GENERAL FORM OF $\dot{K}^\theta{}_\theta$ EQN (36) WHICH MADE USE OF SPECIFIC FORM OF T_{ab} FOR MASSLESS SCALAR FIELD. GENERAL FORM (USEFUL FOR PROJ 3, TERM PROJECTS) IS

$$\boxed{\dot{K}^\theta{}_\theta = \beta K^\theta{}_\theta{}' + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left(\frac{\alpha rb}{a}(rb)'\right)' + \alpha\left(K K^\theta{}_\theta + 4\pi\left(S^r{}_r - \rho\right)\right)} \qquad (36')$$

- POLAR SLICING CONDITION FOLLOWS FROM SETTING $K^\theta{}_\theta = \dot{K}^\theta{}_\theta = K^\theta{}_\theta{}' = 0$, $b = 1$; THEN WE FIND

$$\boxed{\frac{\alpha'}{\alpha} - \frac{a'}{a} + \frac{1-a^2}{r} + 4\pi r a^2\left(\rho - S^r{}_r\right) = 0} \qquad (70)$$

- BUT FROM HAM. C. (62) WE HAVE

$$\frac{a'}{a} = \frac{1-a^2}{2r} + 4\pi r a^2 \rho$$

SO OUR SLICING EQN BECOMES

$$\boxed{\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - 4\pi r a^2 S^r{}_r = 0} \qquad (71)$$

AND FOR THE EMKG SYSTEM, $S^r{}_r = \rho = \frac{1}{2}a^{-2}\left(\Phi^2 + \Pi^2\right)$ SO

$$\boxed{\frac{\alpha'}{\alpha} - \frac{a^2-1}{2r} - 2\pi r\left(\Phi^2 + \Pi^2\right) = 0} \qquad (72)$$

DEFINING $L \equiv \ln \alpha$, WE HAVE

$$\boxed{\begin{aligned} & L' + g_o = 0 \\ & g_o = -\left(\frac{a^2-1}{2r} + 2\pi r \left(\Phi^2 + \Pi^2 \right) \right) \end{aligned}} \qquad (73)$$

AND A SECOND ORDER FD APPROX IS

$$\Delta^r_+ L + \mu^r_+ g_o = 0$$

$$\hookrightarrow \Delta r^{-1} \left(L_{j+1} - L_j \right) + \frac{1}{2} \left(g_{o\,j+1} + g_{o\,j} \right) = 0$$

$$\rightarrow L_j = L_{j+1} + \frac{\Delta r}{2} \left(g_{o\,j+1} + g_{o\,j} \right)$$

WHICH, GIVEN A B.C. AT $r = r_{max}$, CAN EASILY BE INTEGRATED INWARDS (COULD ALSO INT. OUT. FROM $r=0$), ~~ASSUMING~~ $T_{ab} = 0$ AT $r = r_{max}$, AND DEMANDING ~~THAT~~ t MEASURE PROP. TIME FOR COORD. STAT. OBS. AS $r \to \infty$, COMPARISON WITH SCHWARZ. LINE ELEMENT YIELDS

$$\boxed{\alpha(r_{max}, t) = \frac{1}{a(r_{max}, t)}} \qquad (74)$$

EVOLUTION EQUATIONS

GEOMETRY (FROM GEN EQNS (33)-(35))

$$\dot{a} = -\alpha a K^r{}_r \qquad (75)$$

$$K^r{}_r = -\frac{1}{a} \left(\frac{\alpha'}{\alpha} \right)' - \alpha \left(\frac{2}{ar} \left(\frac{1}{r} \right)' + 8\pi \frac{\Phi^2}{a^2} \right) \qquad (76)$$

• RECALLING MOM CONS (69)

$$K^r{}_r = -4\pi r \frac{\Phi \Pi}{a}$$

WE CAN ALSO WRITE

$$\boxed{\dot{a} = 4\pi \alpha r \Phi \Pi} \qquad (77)$$

SCALAR FIELD (FROM GEN. EQNS (37) - (38))

$$\dot{\Phi} = \left(\frac{\alpha}{a} \Pi \right)' \qquad (78)$$

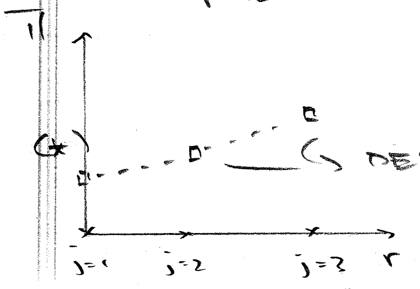
$$\dot{\Pi} = \frac{1}{r^2} \left(r^2 \frac{\alpha}{a} \Phi \right)' = 3 \frac{\partial}{\partial (r^3)} \left(r^2 \frac{\alpha}{a} \Phi \right) \qquad (79)$$

• MOTIVATION FOR $r^{-2} \frac{\partial}{\partial r} \to 3 \frac{\partial}{\partial (r^3)}$: IMPROVED REG. AT $r=0$

$$r^2 \frac{\alpha}{a} \Phi \sim r^3 \text{ AS } r \to 0$$

• DIFFERENCING NOTE: $\dot{\Pi}$ EQN NAIVELY SINGULAR AT $r=0$, COULD REGULARIZE VIA L'HOPITAL'S RULE, ALTERNATE STRATEGY MAKES USE OF

$$\lim_{r \to 0} \Pi(r,t) = \Pi_0(t) + r^2 \Pi_2(t)$$



↳ DETERMINED VIA DISCRETE VERSION of (29)

(*) DETERMINED VIA "QUADRATIC FIT"

$$\boxed{\Pi_1 = \frac{1}{3}\left(4\Pi_2 - \Pi_3\right)}$$

UPDATE SCHEME

- CLEARLY, WOULD BE PERVERSE TO USE $K^r{}_r$ EV. EQN INSTEAD of (ALGEBRAIC) MOM CONS. WHICH EFF. ELIMINATES $K^r{}_r$

- COULD USE $\dot{a}$ EQN RATHER THAN HAM CONS TO UPDATE a, BUT
   a) NEED H.C. SOLVER AT $t=0$ ANYWAY
   b) USING H.C. TENDS TO GIVE IMPROVED STABILITY

FULLY CONSTRAINED SCHEME (SEE ONLINE REF FOR PROJ 2)

$$\dot{\Phi} = \left(\frac{\alpha}{a}\Pi\right)' \qquad \dot{\Pi} = 3\frac{\partial}{\partial(r^3)}\left(r^2\frac{\alpha}{a}\Phi\right)$$

$$\frac{a'}{a} + \frac{a^2-1}{2r} - 2\pi r\left(\Phi^2 + \Pi^2\right) = 0$$

$$\frac{\alpha'}{\alpha} - \frac{a^2-1}{a} - 2\pi r\left(\Phi^2 + \Pi^2\right) = 0$$

NO AH'S IN PA COORDS

$$(rb)' = arbK^\theta{}_\theta \qquad (49)$$

$$\Rightarrow 1 = 0\,!$$

r = const CAN NOT BECOME NULL TIMELIKE BY CONSTRUCTION

SIGNATURE of BH FORMATION: $2m(r,t)/r \to 1$

A) WEAK-FIELD (FLAT SPACETIME) LIMIT

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

I.E. $\alpha(r,t) = a(r,t) \equiv 1$, $\beta(r,t) \equiv 0$

- SCALAR FIELD E.O.M.

$$\Box \phi = 0 \quad \rightarrow \quad \phi_{tt} = \frac{1}{r^2}\left(r^2 \phi_r\right)_r$$

$$\boxed{(r\phi)_{tt} = (r\phi)_{rr}}$$

GENERAL SOLUTION (SCHEMATIC)

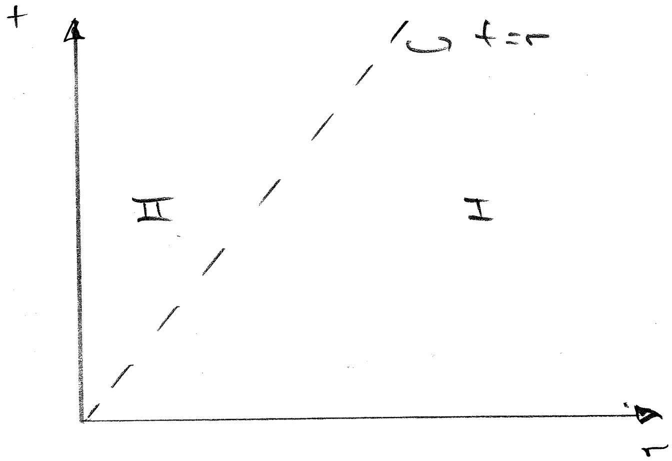
$$(r\phi)(r,t) = \underbrace{f(t+r)}_{\text{INGOING}} + \underbrace{g(t-r)}_{\text{OUTGOING}}$$

- SPECIFY INITIAL DATA IN TERMS OF INITIALLY INGOING PROFILE $f(r)$ AND INITIALLY OUTGOING " $g(r)$

$$(r\phi)(r,0) = f(r) + g(r)$$

$$(r\phi)_t(r,0) = f'(r) - g'(r)$$

COMPLETE SOLUTION



REGION I: $t \leq r$

$$(r\phi)(r,t) = f(t+r) + g(t-r)$$

REGION II: $t > r$ ONLY DATA WHICH WAS INCOMING AT $t=0$ CAN HAVE INFLUENCE HERE. SOL'N MUST STILL BE OF FORM

$$(r\phi)(r,t) = f(t+r) + h(t-r)$$

$$\rightarrow \quad \phi = \frac{f(t+r)}{r} + \frac{h(t-r)}{r}$$

NOW EXAMINE BEHAVIOUR NEAR $r=0$, ASSUME $\phi(0,t)$, f, h SMOOTH, THEN AS $r \to 0$

$$f(t+r) = f(t) + r f'(t) + \tfrac{1}{2} r^2 f''(t) + \dots$$

$$h(t-r) = h(t) - r h'(t) + \tfrac{1}{2} r^2 h''(t) + \dots$$

SO, AS $r \to 0$

$$\phi(t,r) = \frac{f(t)+h(t)}{r} + f'(t) - h'(t) + \frac{1}{2}r\left(f''(t)+h''(t)\right) + \ldots$$

CLEARLY, FOR THE SOL^N TO BE REGULAR AT $r=0$ MUST HAVE

$$h(t) = -f(t) \to h=-f \to h^{(n)} = -f^{(n)}$$

THEN, AS $r \to 0$

$$\phi(r,t) = 2f'(t) + \frac{1}{3}r^2 f'''(t) + \ldots$$

AND WE ALSO RECOVER

$$\phi_r(0,t) = 0$$

(REGULARITY CONDITION) AS PREVIOUSLY

SUMMARY OF COMPLETE (WEAK-FIELD) SOL^N

GIVEN INITIAL DATA
$$(r\phi)(r,0) = f(r) + g(r)$$
$$(r\phi)_t(r,0) = f'(r) - g'(r)$$

$$\boxed{(r\phi)(r,t) = \begin{cases} f(t+r) + g(t-r) & t \le r \\ f(t+r) - f(t-r) & t > r \end{cases}}$$

B) STRONG-FIELD REGIME

- NO GENERAL CLOSED FORM SOL^{N} KNOWN (YET)

INITIAL DATA SPECIFY $\Phi(r,0)$, $\Pi(r,0)$, TYPICALLY BY SPECIFYING $\phi(r,0) = f(r)$ AND ASSUME FLAT-SPACE PROPAGATION ($\Delta t \to 0$) AND INGOING-ONLY DATA

- SOLVE CONSTRAINT ; SLICING EQN^{S} FOR $a(r,0)$, $\alpha(r,0)$

- EVOLVE DATA TO GENERATE S.T.

INTERPOLATING FAMILIES

- CONSIDER PARAMETERIZED FAMILIES OF INITIAL DATA

$$\Phi(r,0;p) \quad \Pi(r,0;p)$$

GENERATES SOL^{N} $S[p]$

EXAMPLE $\quad \phi(r,0) = \phi_0 r^3 \exp\left\{ -\left[\dfrac{r-r_0}{\Delta} \right]^q \right\}$

ANY OF ϕ_0, r_0, Δ, q CAN BE USED AS FAMILY PARAMETER

- HEURISTICALLY AS p DECREASES, GET WEAK FIELD LIMIT, AS P INCREASES, $S[p]$ IS STRONGER-FIELD, EVENTUALLY GET BLACK HOLE FORMATION (SOMETIMES ORIENTATION IS REVERSED, Δ FOR EXAMPLE)
- THESE FAMILIES GENERICALLY "INTERPOLATE" BETWEEN NO-BH AND BH SPACETIMES, BH FORMATION TURNS ON AT CRITICAL VALUE p^*

 PHENOMENOLOGY of CRITICAL COLLAPSE

- INTERPOLATING FAMILIES SIp1 : SEE FIGURE of MASS ASPECT

- FOR GENERIC INTERP. FAMILIES (RECALL: $\Phi(r,0;p)$ $\Pi(r,0;p) \Rightarrow$ "INFINITE DIMENSIONAL" INITIAL-DATA SPACE) ALWAYS FIND THAT

$$\Pi = \ln|p-p^*|$$

IS "NATURAL" PARAMETER FOR DESCRIBING PHENOMENOLOGY of SOLUTION SPACE AS $p \to p^*$

- NEAR-CRITICAL, STRONG-FIELD DYNAMICS (ALWAYS PLAYS OUT IN SOME NEIGHBORHOOD of $r=0$) CHARACTERIZED BY (ESSENTIALLY) UNIQUE SOL^N OF EMKG EQUATIONS

- CONVENIENT TO INTRODUCE NEW SCALAR FIELD VARIABLES

$$X(r,t) \equiv \sqrt{2\pi}\,\frac{r}{a}\,\Phi = \sqrt{2\pi}\,\frac{r}{a}\,\phi_r$$

$$Y(r,t) \equiv \sqrt{2\pi}\,\frac{r}{a}\,\Pi = \sqrt{2\pi}\,\frac{r}{\alpha}\,\phi_t$$

THEN

$$\frac{dm}{dr} = 4\pi r^2 \rho = X^2 + Y^2$$

$$R = -8\pi T = \frac{4}{r^2}\left(X^2 - Y^2\right)$$

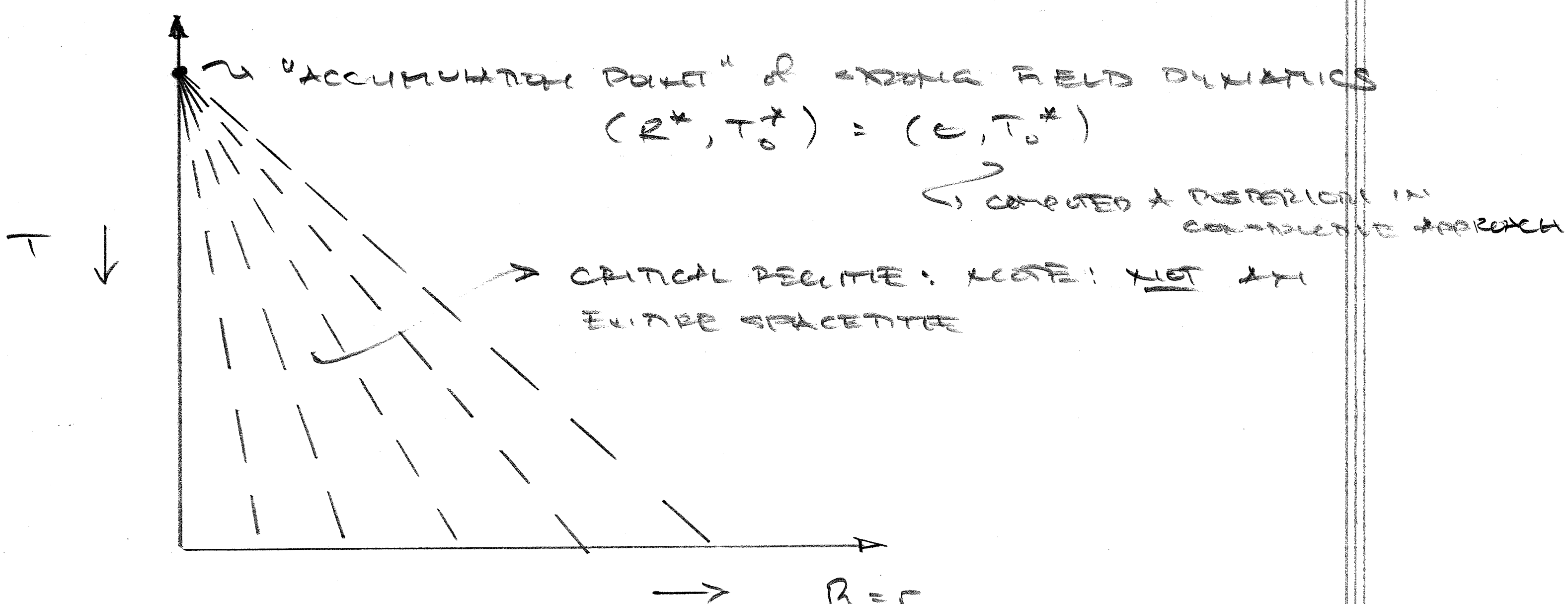
4-D RICCI SCALAR

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2$$

- NATURAL PARAMETERIZATION OF $t =$ CONST HYPERSURFACES FROM P.O.V. OF CRITICAL PHENOMENA → USE PROPER TIME OF CENTRAL OBSERVER (OBS. FIXED AT $r=0$)

$$T_0(t) = \int_0^t \alpha(0,\tilde{t})\, d\tilde{t}$$

- SCHEMATIC STRUCTURE OF CRITICAL SOLUTION



$$T = T_0^* - T_0$$

- DEFINE LOGARITHMIC COORDS

$$\tau = |\ln T|$$

(ASSUME $T, R < 1$)

$$\rho = |\ln R|$$

CRITICAL SOLUTION: DENOTE SCHEM. AS

$$\underline{Z}^*(\rho, \tau)$$

$\underline{Z}^* = \{X^*, Y^*\}$, FOR EXAMPLE, SUFFICES TO COMPLETELY DESCRIBE SOL'N

KEY PROPERTY: DISCRETE SELF-SIMILARITY (DSS) (aka "SCALE-PERIODICITY", "ECHOING")

$$\boxed{\underline{Z}^*(\rho, \tau) \simeq \underline{Z}^*(\rho \pm n\Delta, \tau \pm n\Delta)}$$

IN CONTEXT OF SOL'N GENERATED FROM (ASYMPTOTICALLY FLAT) GENERIC INTERPOLATING FAMILY, MEANS THAT EQUIV. HOLDS

a) IN LIMIT $p \to p^*$

b) IN LIMIT $\rho, \tau \to \infty$ $(R, T) \to (0,0)$

c) ONLY IN STRONG-FIELD CRITICAL REGIME (VALID ONLY WHERE VALID)

$\Delta = 3.44\ldots$ IS UNIVERSAL SCALING EXPONENT

DOES NOT DEPEND ON PARTICULAR INTERPOLATING FAMILY

- DYNAMICS REPEATS ON SCALES RELATED BY FACTOR $e^{\Delta} \simeq 30$ (SEE FIGURE 3)

- PRECISELY CRITICAL SOL'N "ECHOES" INFINITE #

OF TIMES, SINGULAR AT $(R,T) = (0,0)$

- $X, Y, X^2+Y^2, X^2-Y^2, \ldots$ OSCILLATE BETWEEN FIXED, UNIVERSAL LIMITS → SOL^N REMAINS STRONG-FIELD DOWN TO ARBITRARILY SMALL SCALES

$$R = \frac{4}{r^2}(X^2 - Y^2) \to \infty$$

as $r \to 0$

APPARENT HORIZON NEVER FORMS IN PRECISELY-CRITICAL OR SUB-CRITICAL EVOLUTIONS; ALL SCALAR FIELD EVENTUALLY DISPERSES TO INFINITY → CRITICAL SING. IS "NAKED" (VISIBLE TO OBSERVERS AT ∞)

- CRITICAL SOL^N CLEARLY UNSTABLE (BY CONSTRUCTION), INF. PERTURBATIONS WILL RESULT IN EITHER COMPLETE DISPERSAL OR BH FORMATION

- ARBITRARILY SMALL BHs CAN BE PRODUCED VIA FINE-TUNING OF $p - p^*$ → BH TRANSITION IN EMKG MODEL IS TYPE II (ANALOGOUS TO 2ND-ORDER PHASE TRANS. IN STAT MECH SYSTEM, M_{BH} ~ ORDER PARAMETER)

- BLACK HOLE MASS SCALING: FIND

$$\boxed{M_{BH} \simeq C_f \, |p - p^*|^\gamma} \qquad (*)$$

C_f = FAMILY-DEPENDENT CONSTANT

$\gamma = 0.37\ldots$ UNIVERSAL MASS SCALING EXPONENT

- PLAUSIBILITY / CONSISTENCY OF (*)

CONSIDER TWO FAMILIES

$$S[p] \quad , \quad S[q]$$

WHERE THE FAMILY PARAMETERS p, q ARE RELATED BY A SMOOTH TRANSFORMATION

$$q = q(p) \qquad \text{AND} \qquad q^* = q(p^*)$$

THEN (ASSUMING $q > q^*$ IS SUPERCRITICAL REGIME)

$$\lim_{q \to q^*} M_{BH} \sim c_q (q - q^*)^\gamma$$

$$= c_q \left(q(p) - q(p^*) \right)^\gamma$$

$$\sim c_q \left(\left. \frac{dq}{dp} \right|_{p=p^*} (p - p^*) \right)^\gamma$$

$$= c_q \frac{dq}{dp}(p^*)^\gamma (p - p^*)^\gamma$$

$$= c_p (p - p^*)^\gamma$$

- OFTEN GET ASKED, GIVEN A FAMILY $S[p]$ WITH CRIT. VALUE p^*, WHY CAN'T I REDEFINE

$$p \to \tilde{p} = |p - p^*|^\gamma$$

THEN

$$m_{BH} \sim \tilde{p}$$

ANSWER: YOU CAN, BUT FAMILY NOW REQUIRES A PRIORI INFINITE FINE-TUNING; MUST SPECIFY p^* TO ARBITRARY PRECISION; I.E. NEW INTERPOLATING FAMILY IS NOT GENERIC

* CONSTRUCTIVE APPROACH ILLUSTRATES THIS FACT NICELY; IF ONE TRIES TO REPARAM. A FAMILY AS ABOVE, THEN AS $\hat{p} \to \tilde{p}^*$, ANY PERTURBATION (INEVITABLE NUMERICALLY) WILL LEAD TO GENERIC SCALING BEHAVIOUR

* NOTE THAT THIS IS A BASIC OBSERVATION ABOUT "USUAL" CRITICAL PHENOMENA AS WELL, COULD TRY TO PLAY SAME GAME WITH SPIN SYSTEM

$$M \sim |T - T_c|^{\beta}$$

CAN'T (EXPERIMENTALLY) SPECIFY T_c A PRIORI TO ARBITRARY ACCURACY

PHY 387M CRITICAL COLLAPSE: SELF SIMILARITY

REFERENCE: GUNDLACH gr-qc/9712084 ; REFS THEREIN

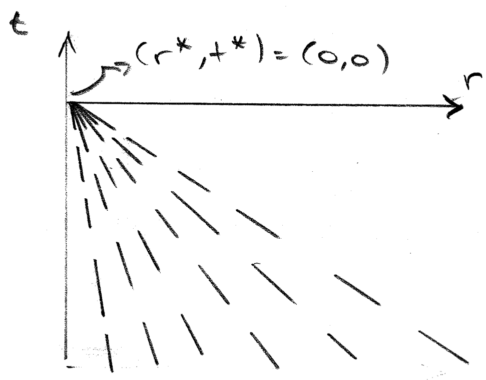
SCHEMATIC PICTURE REVISITED

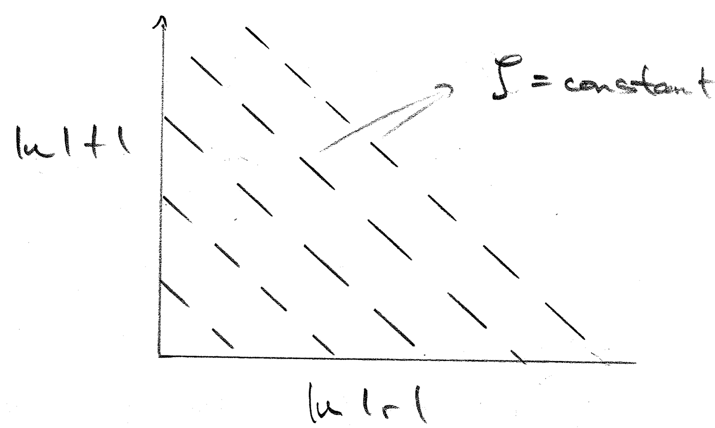
NOTE: IN TODAY'S DISCUSSION WILL (CONTINUE TO) ASSUME

a) SPHERICAL SYMMETRY

b) POLAR/AREAL COORDS WITH $\alpha(0,t) = 1$ $(t \to T_0)$

c) GUNDLACH'S SIGN CONVENTIONS AND VARIOUS NOTATIONS





NATURAL SIMILARITY VARIABLE (COORDINATE)

$$\boxed{\rho \equiv x \equiv -\frac{r}{t}}$$

NEED SECOND COORDINATE, COULD USE EITHER $\tau \sim \ln t$ or $\varsigma \sim \ln r$, FOLLOWING GUNDLACH WILL DEFINE

$$\boxed{\tau \equiv -\ln \frac{t}{L}}$$

$L \to$ IRRELEVANT, FIDUCIAL SPACETIME SCALE

- OBSERVE TWO TYPES of SELF-SIMILARITY IN TYPE II CRITICAL COLLAPSE

A) CONTINUOUS SELF-SIMILARITY (CSS)

$$\boxed{Z^*(x, \tau) = Z^*(x, \tau') = Z^*(x)}$$

OBSERVED IN COLLAPSE of

(i) PERFECT FLUID WITH E.O.S $P = k\rho$ (EVANS ; COLEMAN)

(ii) CERTAIN MULTI-SCALAR MODELS (LIEBLING ; CHOPTUIK)

B). DISCRETE SELF-SIMILARITY (DSS)

$$\boxed{Z^*(x, \tau + \Delta) = Z^*(x, \tau)}$$

"MODEL DEPENDENT ECHOING EXPONENT"

OBSERVED IN COLLAPSE of

(i) SINGLE MASSLESS SCALAR FIELD

(ii) AXISYMMETRIC GRAV WAVES (ABRAHAMS ; EVANS)

(iii) SU(2) YM FIELD (CHOPTUIK, CHMAJ ; BIZON)

- CSS CASE IS EASIER TO ANALYZE - KEY POINT IS THAT CSS ANSATZ REDUCES (IN SPH SYMM) PDE'S TO SET of ODEs + REGULARITY / ANALYTICITY $\Rightarrow$ "EIGENSOLUTIONS" $\equiv$ CRITICAL SOLUTIONS (OR AT LEAST CANDIDATES)

ASIDE: SELF-SIMILARITY IN GR

- WELL STUDIED TOPIC, PARTICULARLY IN SPH SYMM - SEE REFERENCES a) CARR & COLEY b) CARR & HENRIKSEN (TO BE PUBLISHED!) IN GUNDLACH

- HOWEVER, MOST OF THESE STUDIES DO NOT CONSIDER SS FROM "DYNAMICAL PERSPECTIVE" (EXTENT TO WHICH SOLN'S ACT AS "ATTRACTORS" etc.), SO RELEVANCE of SOLN'S TO BH FORMATION-SCENARIOS UNDERAPPRECIATED

COORDINATE-FREE DESCRIPTION of SS IN GR

(SEE WALD APP C, 443-444)

- CONFORMAL KILLING VECTORS

η^a IS A CONFORMAL KILLING VECTOR FIELD (CKV) OF A SPACETIME IF

$$\boxed{\mathcal{L}_\eta g_{ab} = \alpha g_{ab}}$$

FOR SOME FUNCTION α

$$\nabla_a \eta_b + \nabla_b \eta_a = \alpha g_{ab} \qquad (C.3.13)$$

- TAKE TRACE

$$2 \nabla^c \eta_c = n \alpha \quad \Rightarrow \quad \alpha = \frac{2}{n} \nabla^c \eta_c$$

$$\boxed{\nabla_a \eta_b + \nabla_b \eta_a = \frac{2}{n} \nabla^c \eta_c \, g_{ab}} \qquad (C.3.14)$$

- HOMOTHETIC KILLING VECTORS

- SPECIAL CASE OF CKV: ξ^a IS A HOMOTHETIC KV IF

$$\boxed{\begin{aligned} \mathcal{L}_\xi g_{ab} &= 2 g_{ab} \\ \nabla_a \xi_b + \nabla_b \xi_a &= 2 g_{ab} \end{aligned}}$$

NOTE: THE CONSTANT "2" IS CONVENTIONAL AND CAN BE REPLACED WITH AN ARB NON-ZERO VALUE VIA RESCALING OF ξ^a

THEN "SPACETIME HAS A CONTINUOUS SELF-SIMILARITY"

$\equiv$

"SPACETIME HAS A HOMOTHETIC KILLING VECTOR"

$\equiv$

"SPACETIME HAS A HOMOTHETY"

- CAN BE EXTENDED TO DSS CASE (GUNDLACH) BUT MUCH LESS STUDIED

EXERCISE: SHOW THAT IF ξ^a IS A HKV SO THAT $\mathcal{L}_\xi g_{ab} = \lambda g_{ab}$, λ CONSTANT THEN

$$\mathcal{L}_\xi R_{abcd} = \lambda R_{abcd}$$

$$\mathcal{L}_\xi R_{abc}{}^d = 0 \qquad \mathcal{L}_\xi G_{ab} = 0$$

OVERVIEW of ANALYSIS of TYPE II SOL^{N}S

- FOR GIVEN MODEL, WHICH INCLUDES SPEC. of
  - MATTER CONTENT
  - COUPLINGS
  - SYMMETRY

GENERICALLY FIND "ISOLATED" CRITICAL (THRESHOLD) SOLUTION, Z^*

- Z^* CAN BE CONSTRUCTED
  a) INDIRECTLY VIA DYNAMIC EVOLUTION of 1-PARAMETER INTERPOLATING FAMILIES
  b) DIRECTLY FROM ANSATZ REFLECTING PART. SELF-SIMILARITY (CSS or DSS) EXHIBITED BY SOL^N

- ~~CAN~~ THEN DO PERTURBATION THEORY USING Z^* AS BACKGROUND

- ASSUME Z^* IS CSS, THEN FOR A NEAR-CRITICAL SOL^N $Z(x,\tau)$, WE WILL HAVE, TO LINEAR ORDER

$$\boxed{\begin{aligned}\delta Z(x,\tau) &= Z(x,\tau) - Z^*(x,\tau)\\ &= \sum_{i=1}^{\infty} c_i\, e^{\lambda_i \tau} f_i(x)\end{aligned}}$$

$f_i(x)$: EIGENMODES

λ_i : CORRESPONDING EIGENVALUES

c_i : COEFFICIENTS

- CRUCIAL OBSERVATION CONCERNING TYPE II CRITICAL SOL^N WAS MADE BY KOIKE et al (PRL, 74, 5170 (1995)) "SHARPNESS" of PHASE TRANSITION SUGGESTED THAT ONLY ONE MODE $f_1(x)$ WAS UNSTABLE; I.E. HAD AN E.V. λ_1 WITH $\operatorname{Re}\lambda_1 > 0$ (EVANS ALSO HAD THE IDEA)

- THUS TYPE II CRITICAL SOL^N $\equiv$ MINIMALLY UNSTABLE SELF-SIMILAR SOL^N

- PROCEEDING ON THIS BASIS, WE THEN HAVE AS $p \to p^*$

$$Z(x,\tau;p) \simeq Z^*(x,\tau) + C_1(p)\, e^{\lambda_1 \tau} f_1(x)$$

$$C_1(p^*) = 0 \Rightarrow C_1(p) \simeq \left.\frac{dC_1}{dp}\right|_{p^*} (p-p^*) \equiv \frac{dC_1}{dp}(p-p^*)$$

$$\simeq Z^*(x,\tau) + \frac{dC_1}{dp}(p-p^*)\, e^{\lambda_1 \tau} f_1(x)$$

BLACK HOLE MASS SCALING (ASSUME THAT $\operatorname{Re}\lambda_1 = \lambda_1$)

- AS $p \to p^*$, $Z(x,\tau;p) \simeq Z^*(x,\tau)$ FOR MORE & MORE DECADES of SCALE, DEV. δZ GOVERNED BY $f_1(x)$ (ONE SIGN $\to$ DISPERSAL; OTHER SIGN $\to$ BH FORMATION)

- HEURISTICALLY, BH FORMATION WILL OCCUR WHEN $(p-p^*)\, e^{\lambda_1 \tau_p}$ REACHES SOME CRITICAL VALUE (INDEPENDENT of SCALE, τ_p) AND ALWAYS AT THAT SCALE τ_p

RECALL: $\tau \equiv -\ln \frac{t}{L} \Rightarrow t_p = L\, e^{-\tau_p} \simeq M_{BH}(p)$

THUS, CONDITION FOR BH FORMATION IS

$$(p-p^*)\, e^{\lambda_1 \tau_p} \simeq \text{const.} = K$$

$$\rightarrow \quad \ln(p-p^*) = -\lambda_1 \tau_p + \underbrace{\kappa}_{\ln K}$$

$$= \lambda_1 \ln \frac{t_p}{L} + \kappa$$

$$\rightarrow \quad \ln M_{BH}(p) = \lambda_1^{-1} \ln(p-p^*) + \kappa'$$

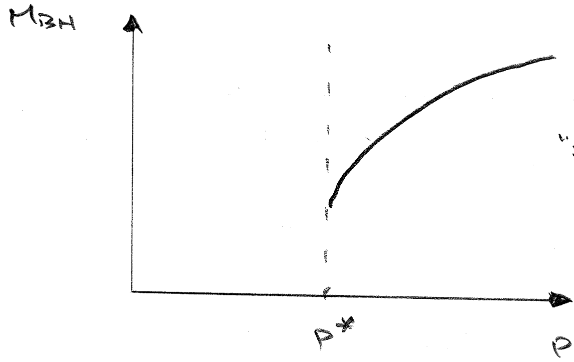
$$\boxed{M_{BH}(p) = c_f\, |p-p^*|^{1/\lambda_1}}$$

WHICH IS THE EMPIRICALLY DEDUCED SCALING LAW WITH

$$\boxed{\gamma = (\mathrm{Re}\, \lambda_1)^{-1}}$$

CRITICAL COLLAPSE: TYPE I SOL^{N}S

- RECALL: BY DEFN, WILL HAVE FOLLOWING BEHAVIOUR OF $M_{BH}(p)$ FOR TYPE I SOLNS:



"FIRST-ORDER PHASE TRANS."

- SO FAR, TYPE I SOLN^S ARE STATIC / QUASI-STATIC (PERIODIC) SOL^{N}S TO COUPLED EINSTEIN / MATTER EQN^S IN SPH. SYMM. RESTRICT DISCUSSION TO EXACTLY STATIC CASE

- EXAMPLE: SU(2) EINSTEIN-YANG-MILLS (EYM) (CHOPTUIK et al, PRL, 77, 424)

$$ds^2 = -\alpha^2(r,t)\,dt^2 + a^2(r,t)\,dr^2 + r^2 d\Omega^2$$

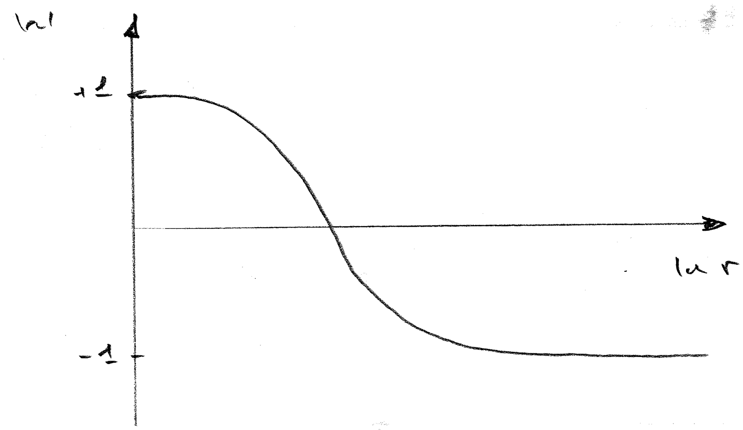
$$L_M = -\left(\frac{\nabla^\mu W \nabla_\mu W}{r^2} + \frac{1}{2}\frac{(1-W^2)^2}{r^4} \right)$$

$W(r,t) \equiv$ YANG-MILLS POTENTIAL

- STATIC SOLN^S: (BARTNIK ; MCKINNON, PRL, 61, 141)

- ASSUME $\alpha = \alpha(r)$, $a = a(r)$, $W = W(r)$ - LOOK FOR SOL^{N}S OF EYM EQN'S WHICH ARE REGULAR AT $r=0$ AND $r \to \infty$; IN PARTICULAR, MUST HAVE $W(0) = \pm 1$, $W(\infty) = \pm 1$

- FIND COUNTABLE INFINITY of STATIC SOL^NS $W_n(r)$, $n = 1, 2, 3 \ldots$ WHERE n COUNTS # of 0-CROSSINGS OF $W(r)$



- SUBSEQUENT STUDY SHOWED THAT ALL THE $W_n(r)$ STATIC SOL^NS ARE UNSTABLE WITH PRECISELY n UNSTABLE MODES IN PERT THEORY WITHIN PARTICULAR ANSATZ ("PURELY MAGNETIC")

- SUGGESTED TO BIZON THAT $W_1(r)$ MIGHT BE CRITICAL SOL^N IN SENSE WE HAVE BEEN DISCUSSING IT

- EXPECTATION BORNE OUT BY "USUAL" GAME INVOLVING 1-PARAMETER INTERPOLATING FAMILIES GENERATED FROM INITIAL DATA $W(r, 0; p)$

NOTE: $W(r, 0; p)$ DOES NOT NECESSARILY HAVE TO BE A "KINK" IN ORDER FOR $W(r, 0; p^*) \simeq W_1(r)$

- PHENOMENOLOGY

$$\text{AS } p \to p^*, \quad W(r, t) \to W_1(r)$$

- SCALING LAW: "LIFETIME", τ, OF CONFIGURATION

$$\boxed{\tau \sim -\sigma \ln |p - p^*|} \qquad \sigma: \text{UNIVERSAL EXPONENT}$$

- PERTURBATION THEORY: THIS TIME, IT WAS ALREADY KNOWN THAT (AGAIN, WITHIN PART. SELF-CONSISTENT ANSATZ), $u_1(r)$ HAD ONLY ONE UNSTABLE MODE

$$Z(r,t;p) \simeq Z^*(r,t) + C_1(p)\, e^{\lambda_1 t} f_1(r)$$

$$\simeq Z^*(r,t) + \left.\frac{dC_1}{dp}\right|_{p^*} \underbrace{(p-p^*)\, e^{\lambda_1 t}} f_1(r)$$

- USING THE SAME ARG. AS FOR TYPE II CASE, BH FORMATION (OR DISPERSAL) SIGNALLED BY

$$(p-p^*)\, e^{\lambda_1 t_p} = K$$

$$e^{\ln(p-p^*)}\, e^{\lambda_1 t_p} = K$$

$$\Rightarrow \quad \boxed{t_p = -\frac{1}{\lambda_1} \ln(p-p^*)} \qquad (t_p = \tau)$$

SO $\quad \boxed{\sigma = \frac{1}{\lambda_1}}$

- NOTE: MARGINALLY SUPER-CRITICAL EVOLUTIONS

$$M_{BH} \simeq M_1 \equiv \text{TOTAL MASS (ADM MASS) OF STATIC } n=1 \text{ CONFIGURATION}$$

• MEASUREMENT OF LIFETIME: AGAIN, CRUCIAL OBSERVATION IS THAT FOR $p_1 \simeq p^*$, $p_2 \simeq p^*$ $|p_1 - p^*| > |p_2 - p^*|$, LATE-TIME p_2 EVOLUTION WILL QUITE PRECISELY MATCH LATE-TIME p_1 EVOLUTION (DEPARTURE FROM CRITICALITY IS "UNIVERSAL" - I.E. DESCRIBED BY $e^{\lambda_\ell t} f_\ell(r)$) PROVIDED WE SHIFT p_2 EVOLUTION IN TIME TO MATCH RESPECTIVE PEEL-OFF TIMES. THUS

$$\sigma \simeq \frac{\delta t_p}{\delta \ln(p-p^*)}$$

CAN BE COMPUTED BY LOOKING ~~AT~~ SOME FEATURE IN THE NON-CRITICAL (I.E. LATE-TIME) REGIME. FOR EXAMPLE, DEFINE

$$T_r(\ln(p-p^*)) \quad \text{FOR SUB-CRITICAL EVOLUTIONS}$$

TO BE THE CENTRAL PROPER TIME AT WHICH THE 0-CROSSING OF W ARRIVES AT RADIUS r AS IT PROPAGATES OUTWARD (I.E. $W(r, T_r(\ln(p-p^*))) = 0$, THEN AS $p \to p^*$

$$-\frac{dT_r}{d\ln(p-p^*)} \to \sigma$$

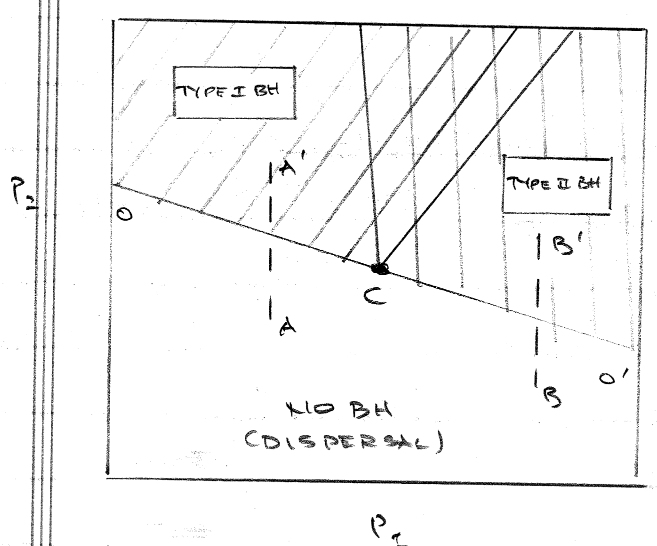
TYPICAL INTERP. FAMILY: $\sigma \sim 0.55(2)$

PERTURBATION THEORY: $\sigma = 0.5519\ldots$

2-PARAMETER FAMILIES AND TYPE I ↔ TYPE II TRANS.

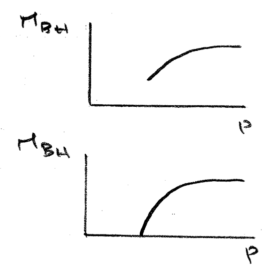
- SUITABLE 2-PARAM. FAMILIES OF INITIAL DATA $W(r,0;p_1,p_2)$ CAN GENERATE BOTH TYPE I AND TYPE II BEHAVIOUR

"PHASE DIAGRAM"



OO': CRITICAL LINE

AA'



BB'

C: CO-EXISTENCE POINT (NAKED SINGULARITY INSIDE $n=1$ B-MK SOLUTION)

- EXAMPLE: EINSTEIN-MASSIVE-KLEIN-GORDON COLLAPSE (BRADY et al, PRD, 56, 6057)

$$\Box\phi(r,t) = m^2\phi(r,t)$$

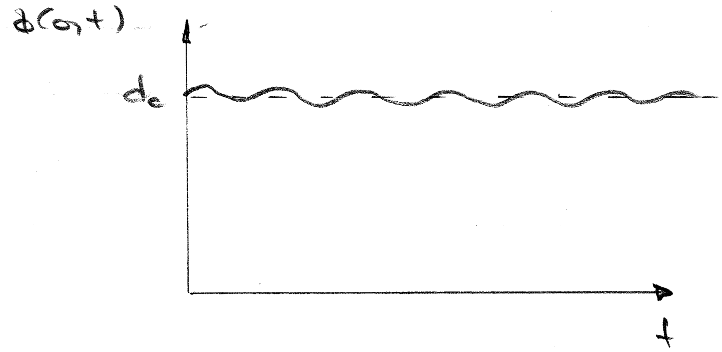
$$L_M = -\left(\nabla^\mu\phi\nabla_\mu\phi + \tfrac{1}{2}m^2\phi^2\right)$$

- SEIDEL & SUEN (PRL, 66, 1659) HAD PREVIOUSLY DISCOVERED PERIODIC (QUASI-STATIC) SOLN'S TO EKG EQUATIONS WHICH THEY DUBBED "OSCILLON STARS"

$$g(r,t) \sim \sum_{n=0}^{\infty} e^{in\omega t} f_n(r)$$

("EIGENFREQUENCY")

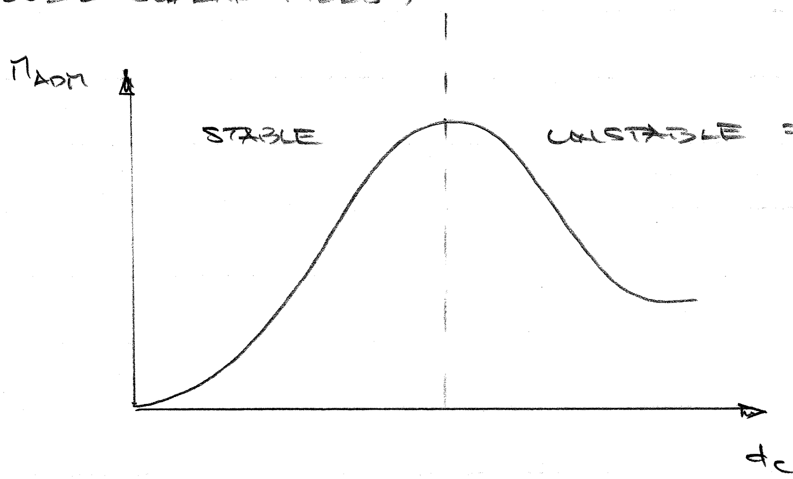
WITH $|f_0(r)| \gg |f_n(r)| \quad n > 0$



INCORRECT!

ϕ HAS NO "DC" COMPONENT

- THESE OSCILLATONS FORM A 1-PARAMETER FAMILY, LABELED, FOR EXAMPLE BY $\phi(0,t) \equiv \phi_c$ (MEAN CENTRAL VALUE of SCALAR FIELD)



UNSTABLE ⇒ TYPE I CRITICAL SOLNS

- THIS TIME THERE IS A CONTINUOUS FAMILY of TYPE I CRIT. SOLNS (NOT JUST 1 AS IN EYM CASE)

- TYPE II BEHAVIOUR → TYPE II CRIT. SOLN PRECISELY THE SAME AS Z^* FOR EMKG → FIELD NATURALLY GETS DRIVEN TO MASSLESS ≡ SCALE FREE ≡ KINETIC-ENERGY-DOMINATED STATE AS $p \to p^*$

$$\left.\begin{array}{l} \phi^* \text{ BOUNDED} \Rightarrow V(\phi^*) = \tfrac{1}{2} m^2 \phi^{*2} \text{ BOUNDED} \\ \nabla^\mu \phi \nabla_\mu \phi \text{ UNBOUNDED} \Rightarrow \text{K.E. DOMINATES} \end{array}\right\} \to \text{TYPE OF UV ASYMPTOTIC FREEDOM}$$